多重ゼータ関数の解析接続とその性質

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色数

お品書き

1. ゼータ関数について
2. 多重ゼータ値について
3. 多重ゼータ関数について
4. 私の得た結果について
5. 展望

Notation

Definition

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\cdots(\alpha+n-1)$$

Riemann のゼータ関数

Theorem (Euler)

$$\zeta(2) = \frac{\pi^2}{6}$$

Riemann のゼータ関数

Theorem (積分表示)

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt$$

証明のスケッチ

$$\Gamma(s) \coloneqq \int_0^\infty e^{-t} t^{s-1} dt$$

Riemann のゼータ関数

Theorem (負の整数点)

$n \geq 0$ を満たす整数 n に対して,

$$\zeta(-n) = \frac{(-1)^n B_{n+1}}{n+1}$$

が成り立つ.

証明のスケッチ

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{(-z)^{s-1}}{e^z - 1} dz$$

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

Riemann のゼータ関数

Riemann Hypothesis

$\zeta(s)$ の非自明な零点はすべて $\Re(s) = \frac{1}{2}$ 上にある.

Definition (多重ゼータ値)

正整数 r に対し, r 個の正整数の組 $\mathbf{k} = (k_1, \ldots, k_r)$ をインデックス, とくに $k_r > 1$ であるものを収束インデックスと呼ぶ. 多重ゼータ値は収束インデックス $\boldsymbol{k} = (k_1, \ldots, k_r)$ に対し次の多重級数で定義される.

$$\zeta(k_1, \ldots, k_r) \coloneqq \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

Hoffman 代数

Definition (Hoffman 代数)

有理数係数の 2 変数非可換多項式環 $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$ に対し部分代数 $\mathfrak{H}^1$, $\mathfrak{H}^0$ を

$$\mathfrak{H}^1 := \mathbb{Q} + y\mathfrak{H}, \quad \mathfrak{H}^0 := \mathbb{Q} + y\mathfrak{H}x$$

で定める．また, $\mathbb{Q}$-線形写像 $Z : \mathfrak{H}^0 \to \mathbb{R}$ を

$$Z(1) = 1, \quad Z(yx^{k_1-1}yx^{k_2-1}\cdots yx^{k_r-1}) = \zeta(k_1, \ldots, k_r)$$

で定める．

多重ゼータ値の満たす関係式

双対性

$\mathfrak{H}$ の反自己同型写像 τ を $\tau(x) = y$, $\tau(y) = x$ で定めたとき，任意の $w \in \mathfrak{H}^0$ に対し $\tau(w) - w \in \mathrm{Ker} Z$ が成り立つ．
例えば，$\zeta(3) = \zeta(1, 2)$, $\zeta(3, 1, 4) = \zeta(1, 1, 3, 1, 2)$ など．

和公式

正整数 k, r に対し

$$\sum_{\boldsymbol{k} \in I_0(k,r)} \zeta(\boldsymbol{k}) = \zeta(k)$$

が成り立つ．例えば，$\zeta(1, 3) + \zeta(2, 2) = \zeta(4)$ など．

多重ゼータ値の反復積分表示

多重ゼータ値を扱う上で Kontsevich によって与えられた次のような事実が知られている.

Theorem (多重ゼータ値の反復積分表示)

微分形式 ω_0, ω_1 を $\omega_0(t) = \dfrac{dt}{t}$, $\omega_1(t) = \dfrac{dt}{1-t}$ と定めたとき, 正整数 r , $\epsilon_1, \dots, \epsilon_r \in \{0, 1\}$ に対し

$$I(\epsilon_1, \dots, \epsilon_r) := \int_{0<t_1<\cdots<t_r<1} \prod_{i=1}^{r} \omega_{\epsilon_i}(t_i)$$

とする. このとき, 収束インデックス $\mathbf{k} = (k_1, \dots, k_r)$ に対し

$$\zeta(\mathbf{k}) = I(1, \{0\}^{k_1-1}, \dots, 1, \{0\}^{k_r-1})$$

が成り立つ. ここで $\{0\}^N$ は 0 を N 個並べた列を指す.

特殊値公式

Theorem (Euler)

正整数 k に対し

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}B_{2k}}{2(2k)!}$$

が成り立つ. ここで B_n は *Bernoulli* 数とした.

Theorem (Broadhurst)

$$\zeta(\{2\}^r) = \frac{\pi^{2r}}{(2r+1)!}, \quad \zeta(\{4\}^r) = \frac{2^{2r+1}\pi^{4r}}{(4r+2)!}$$

次元予想

Zagier 予想

2 以上の整数 k に対し $\mathcal{Z}_k := \mathrm{span}_{\mathbb{Q}}\{\zeta(\boldsymbol{k}) \mid \mathrm{wt}(\boldsymbol{k}) = k\}$ で定義する．
$d_0 = 1,\ d_1 = 0,\ d_2 = 1,\ d_{k+2} = d_{k+1} + d_k \quad (k \geq 0)$
$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ が成り立つだろう．

様々な多重ゼータ関数

Definition

$$\zeta_{EZ,r}(s_1, s_2, \dots, s_r) := \sum_{m_1, m_2, \dots, m_r = 1}^{\infty} m_1^{-s_1}(m_1+m_2)^{-s_2}\cdots(m_1+m_2+\cdots+m_r)^{-s_r}$$

$$\zeta_{MT,r}(s_1, s_2 \dots, s_r; s_{r+1}) := \sum_{m_1, m_2, \dots, m_r = 1}^{\infty} m_1^{-s_1} m_2^{-s_2}\cdots m_r^{-s_r}(m_1+m_2+\cdots+m_r)^{-s_{r+1}}$$

$$\zeta_{AV,r}(s_1, s_2, \dots, s_r; s_{r+1}) := \sum_{0<m_1<m_2<\cdots<m_r} m_1^{-s_1} m_2^{-s_2}\cdots m_r^{-s_r}(m_1+\cdots+m_r)^{-s_{r+1}}$$

$$\zeta_{B,r}(s, \alpha | w_1, \dots, w_r) := \sum_{m_1, \dots, m_r = 0}^{\infty} (\alpha + m_1 w_1 + \cdots + m_r w_r)^{-s}$$

ルート系のゼータ関数

Definition

$$\zeta(\boldsymbol{s};\Delta) := \sum_{m_1,\ldots,m_r=1}^{\infty} \prod_{\alpha\in\Delta_+} \langle \alpha^\vee, m_1\lambda_1 + \cdots + m_r\lambda_r \rangle^{-s_\alpha}$$

ルート系のゼータ関数

例

$$\zeta(\mathbf{s}; A_2) = \sum_{m_1, m_2 = 1}^{\infty} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3}$$

$$\zeta(\mathbf{s}; C_2) = \sum_{m_1, m_2 = 1}^{\infty} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3} (m_1 + 2m_2)^{-s_4}$$

絶対収束域 (ゼータ関数)

Theorem

$\zeta(s)$ は $\Re s > 1$ で絶対収束する.

証明のスケッチ

正の整数 N と実数 $\sigma > 1$ に対して

$$\sum_{N<n} \frac{1}{n^\sigma} \leq \frac{N^{1-\sigma}}{\sigma - 1}$$

が成り立つ.

絶対収束域 (多重ゼータ関数)

Theorem (Matsumoto)

$\zeta_{EZ,r}(s_1,\ldots,s_r)$ は $1 \le k \le r$ に対し, $\Re(s_{r-k+1}+\cdots+s_r) > k$ で絶対収束する.

証明のスケッチ

$$\zeta_{EZ,r}(s_1,\ldots,s_r) = \sum_{0<m_1<\cdots<m_{r-1}} \frac{1}{m_1^{\sigma_1}\cdots m_{r-1}^{\sigma_{r-1}}} \sum_{m_{r-1}<m_r} \frac{1}{m_r^{\sigma_r}}$$

積分表示

Theorem

$$\zeta_{EZ,r}(s_1,\ldots,s_r)=\frac{1}{\Gamma(s_1)\cdots\Gamma(s_r)}\int_0^\infty\frac{t_r^{s_r-1}}{e^{t_r}-1}\cdots\int_0^\infty\frac{t_1^{s_1-1}}{e^{t_1+\cdots+t_r}-1}dt_1\cdots dt_r$$

具体例

$$\zeta_{EZ,2}(s_1,s_2)=\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{t_2^{s_2-1}}{e^{t_2}-1}\int_0^\infty\frac{t_1^{s_1-1}}{e^{t_1+t_2}-1}dt_1dt_2$$

解析接続の結果

Theorem

$\zeta_{EZ,r}(s_1,\ldots,s_r)$ は $\mathbb{C}^r$ 全体に有理型函数として解析接続される．また，極は

$$s_r = 1,\ s_{r-1}+s_r = 2,1,0,-1,-2,-4\ldots$$

と

$$\sum_{i=1}^{j} s_{r-i+1} \in \mathbb{Z}_{\le j}\ , 3 \le j \le r$$

で与えられ，すべて 1 位である．

解析接続 (Euler–Maclaurin 和公式)

Theorem (Euler–Maclaurin 和公式)

任意の正整数 n, 区間 $[0, n]$ で C^{2m} 級である関数 $f(x)$ に対し,

$$\sum_{k=1}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}\{f(n) - f(0)\} + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\{f^{(2k-1)}(n) - f^{(2k-1)}(0)\} \\ - \frac{1}{(2m)!}\int_0^n B_{2m}(x - \lfloor x \rfloor) f^{(2m)}(x)dx$$

が成り立つ.

解析接続 (Euler–Maclaurin 和公式)

Akiyama–Egami–Tanigawa

$$\zeta_{EZ,2}(s_1, s_2) = \frac{\zeta_{EZ,1}(s_1+s_2-1)}{s_2-1} - \frac{\zeta_{EZ,1}(s_1+s_2)}{2} + \sum_{q=1}^{\ell} \frac{B_{q+1}(s_2)_q}{(q+1)!}\zeta(s_1+s_2+q) - \sum_{n_1=1}^{\infty} \frac{\phi_\ell(n_1, s_2)}{n_1^{s_1}}$$

解析接続 (Euler–Maclaurin 和公式)

証明のスケッチ

$$\phi_\ell(m,s) = \sum_{n=1}^{m} \frac{1}{n^s} - \left\{ \frac{m^{1-s}-1}{1-s} + \frac{1}{2m^s} - \sum_{q=1}^{\ell} \frac{(s)_q B_{q+1}}{(q+1)!m^{s+q}} + \zeta(s) - \frac{1}{s-1} \right\}$$

が成り立つ．ただし，$\phi_\ell(m,s)$ は $|(s)_{\ell+1}|m^{-\Re(s)-\ell-1}$ で上から評価できる．
また，定義より

$$\zeta_{EZ,2}(s_1,s_2) = \sum_{n_1=1}^{\infty} \frac{1}{n_1^{s_1}} \sum_{n_2=n_2+1}^{\infty} \frac{1}{n_2^{s_2}}$$

が成り立つ．

負の整数点

Definition

$$\zeta_{EZ,r}(-s_1,\ldots,s_r) := \lim_{k_1\to -s_1}\cdots\lim_{k_r\to -s_r}\zeta_{EZ,r}(k_1,\ldots,k_r)$$

Theorem

$$\zeta_{EZ,r}(-s_1,\ldots,-s_r) = \sum_{\ell=-1}^{s_r}\frac{(-r_k)_\ell B_{\ell+1}}{(\ell+1)!}\zeta_{EZ,r-1}(-s_1,\ldots,-s_{r-1}-s_r+\ell)$$

具体例

$$\zeta_{EZ,2}(0,0) = \frac{1}{3}$$

$$\zeta_{EZ,3}(0,0,0) = -\frac{1}{4}$$

Theorem

$$\frac{\zeta_{EZ,1}(-4r-1)}{\zeta_{EZ,2}(-2r,-2r)} = (2r+1)\binom{4r+2}{2r+1}$$

解析接続 (Mellin–Barnes 積分)

導出のスケッチ

$$\Gamma(z)(1+t)^{-z} = \frac{1}{2\pi i}\int_{(c)} \Gamma(z+s)\Gamma(-s)t^s ds$$

を $\zeta_{EZ,2}(s_1, s_2)$ に適用すると,

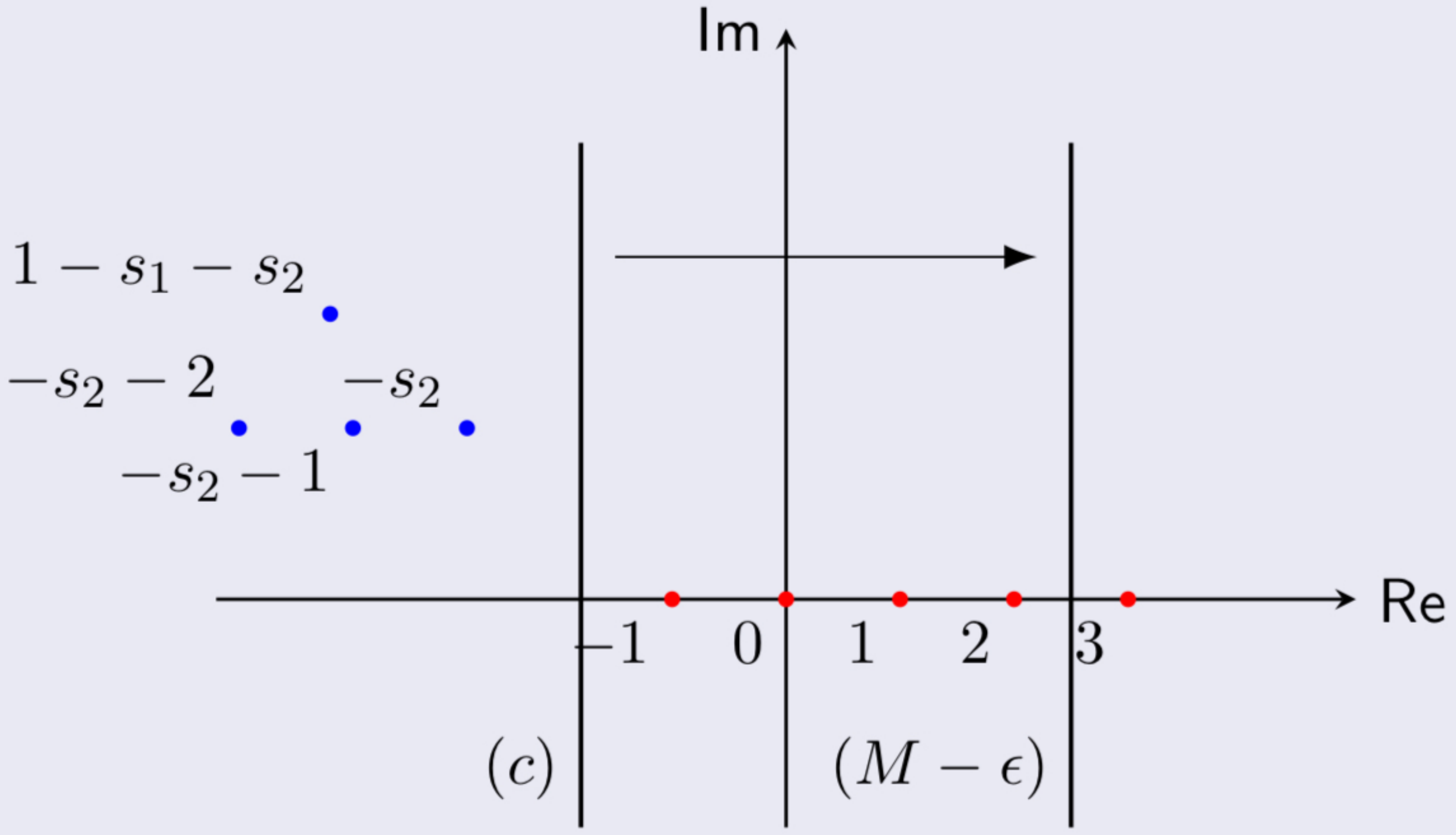
$$\zeta_{EZ,2}(s_1, s_2) = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_2+t)\Gamma(-t)}{\Gamma(s_2)}\zeta(-t)\zeta(s_1+s_2+t)dt$$

を得る. (ここで和を取る際に $\Re u > 1$, $\Re v > 1$ という条件を課している)

解析接続 (Mellin–Barnes 積分)

極の位置

最後の項の被積分関数の極は下図の青い点と赤い点に位置する．ここで積分経路は $c - i\infty$ から $c + i\infty$ への垂直な直線としたが，赤い点に位置する極を右側へ，青い点に位置する極を左側へ分断するように実数 c を取る必要がある．



関数関係式

Theorem (Matsumoto–Tsumura)

$$2\zeta_{MT,2}(1,s,1) - \zeta_{MT,2}(1,1,s) = 2\zeta(s+2)$$

具体例

$s = 1$ として,

$$\frac{1}{mn} = \frac{1}{m+n}\left(\frac{1}{n} + \frac{1}{m}\right)$$

を用いると $\zeta(1,2) = \zeta(3)$ を得る.

関数関係式

Theorem (Hirose–Murahara–Onozuka)

$$\sum_{k=0}^{\infty}(\zeta_{EZ,2}(s-k-2,k+2)-\zeta_{EZ,2}(-k,s+k))=\zeta_{EZ,1}(s)$$

具体例

$s=3$ とすれば $\zeta_{EZ,2}(1,2)=\zeta_{EZ,1}(3)$ を得る.

関数等式 ($r = 1$)

Theorem (Riemann (1859))

$$\zeta_{EZ,1}(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta_{EZ,1}(1-s)$$

が成り立つ.

関数等式 $(r=2)$

Theorem (Matsumoto (2004))

$$\frac{g(u,v)}{(2\pi)^{u+v-1}\Gamma(1-u)} = \frac{g(1-v,1-u)}{i^{u+v-1}\Gamma(v)} + 2i\sin\left(\frac{\pi}{2}(u+v-1)\right)F_{+}(u,v),$$

が成り立つ．ここで

$$F_{+}(u,v) \coloneqq \sum_{k=1}^{\infty}\sigma_{u+v-1}(k)\Psi(v,u+v;2\pi ik),$$

$$g(u,v) \coloneqq \zeta_{EZ,2}(u,v) - \frac{\Gamma(1-u)}{\Gamma(v)}\Gamma(u+v-1)\zeta_{EZ,1}(u+v-1)$$

とした．また，合流型超幾何関数を次で定める．

$$\Psi(a,c;x) \coloneqq \frac{1}{\Gamma(a)}\int_{0}^{\infty e^{i\phi}} e^{-xy}y^{a-1}(1+y)^{c-a-1}dy.$$

関数等式 $(r=2)$

系

$$\Omega_{2k+1} := \{(s_1, s_2) \in \mathbb{C}^2 \mid s_1 + s_2 = 2k+1\} \quad (k \in \mathbb{Z}),$$

という超平面上においては

$$\frac{1}{(2\pi)^{2k}\Gamma(1-s_1)}\zeta_{EZ,2}(s_1, s_2) = \frac{(-1)^k}{\Gamma(s_2)}\left\{\zeta_{EZ,2}(1-s_2, 1-s_1) - \frac{B_{2k}}{4k}\right\}$$

という美しい等式が得られる.

関数等式 ($r = 2$)

証明のスケッチ

合流型超幾何関数は

$$\Psi(a, c; x) = x^{1-c}\Psi(a - c + 1, 2 - c; x)$$

という変換公式を持つ.

関数等式

Theorem (Yokoi (2025))

Rough な主張: r を 2 以上の正整数とする. r 重の *Euler–Zagier* 型多重ゼータ関数 $\zeta_{EZ,r}(s_1, s_2, s_3 \ldots, s_r)$ は $\zeta_{EZ,r}(s_1, s_2, s_3, \ldots, s_r) \longleftrightarrow \zeta_{EZ,r}(1 - \mathrm{wt}(\mathbf{s}) + s_1, 1 - \mathrm{wt}(\mathbf{s}) + s_2, s_3, \ldots, s_r)$ という対称性を持つ.

上の定理は Matsumoto (2004) の結果を含む.

関数等式

Definition (多重合流型超幾何関数)

多重合流型超幾何関数を

$$\Psi_a(h_1, \dots, h_{a+1}; x_1, \dots, x_a; \delta)$$
$$:= \frac{1}{\Gamma(h_2)\cdots\Gamma(h_{a+1})} \int_0^{\infty e^{i\phi}} e^{-x_a t_a} t_a^{h_{a+1}-1} \int_0^{\infty e^{i\phi}} e^{-x_{a-1}t_{a-1}} t_{a-1}^{h_a - 1}$$
$$\times \int_0^{\infty e^{i\phi}} \cdots \int_0^{\infty e^{i\phi}} e^{-x_2 t_2} t_2^{h_3-1} \int_0^{\infty e^{i\phi}} e^{-x_1 t_1} t_1^{h_2-1} (\delta + t_1 + t_2 + \cdots + t_a)^{h_1 - 1}$$
$$\times\, dt_1 \cdots dt_a$$

で定める.

関数等式

証明のスケッチ

a を正整数とする．また，複素数 $h_1, \ldots, h_{a+1}$ は $h_1 < 1$, $0 < h_2 + \cdots + h_{a+1}$ を満たすとする．このとき

$$\begin{aligned}
&\Psi_a(h_1, \ldots, h_{a+1}; x_1, \ldots, x_a; 1) \\
&= x_1^{h_3+\cdots+h_{a+1}} x_2^{-h_3} \cdots x_a^{-h_{a+1}} \\
&\qquad \times \sum_{m_1=0}^{\infty} \frac{(h_3)_{m_1}(1-\frac{x_1}{x_2})^{m_1}}{m_1!} \cdots \sum_{m_{a-1}=0}^{\infty} \frac{(h_{a+1})_{m_{a-1}}(1-\frac{x_1}{x_a})^{m_{a-1}}}{m_{a-1}!} \\
&\qquad \times (1-h_1)_{m_1+\cdots+m_{a-1}} \Psi(h_2+\cdots+h_{a+1}+m_1+\cdots+m_{a-1}, h_1+\cdots+h_{a+1}; x_1).
\end{aligned}$$

が成り立つ．

Coffee break

Riemann ゼータ関数と同じように Euler–Zagier 型多重ゼータ値 (の $r=2$ の場合において) は次のような積分表示

$$\zeta_{EZ,2}(s_1,s_2)=\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^y-1}\int_0^\infty\frac{x^{s_1-1}}{e^{x+y}-1}dxdy \tag{10}$$

を持つ．右辺の積分は $\Re s_1>0$, $\Re s_2>1$, $\Re(s_1+s_2)>2$ という領域において収束する．ここで,

$$h(z):=\frac{1}{e^z-1}-\frac{1}{z}$$

とおく．$h(z)$ を用いて式 (10) の右辺を分割すると,

$$\begin{aligned}(10)=&\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^y-1}\int_0^\infty\frac{x^{s_1-1}}{x+y}dxdy\\&+\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^y-1}\int_0^\infty x^{s_1-1}h(x+y)dxdy\end{aligned} \tag{11}$$

となる．ここで, $B(x,y)$ をベータ関数とすると,

$$B(x,y)=\int_0^\infty\frac{t^{x-1}}{(1+t)^{x+y}}dt \tag{12}$$

が成り立つことを用いると,

$$\begin{aligned}\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^y-1}\int_0^\infty\frac{x^{s_1-1}}{x+y}dxdy&=\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_2-1}}{e^y-1}\frac{1}{y}\int_0^\infty\frac{x^{s_1-1}}{1+\frac{x}{y}}dxdy\\&=\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty\frac{y^{s_1+s_2-2}}{e^y-1}\int_0^\infty\frac{u^{s_1-1}}{1+u}dudy\\&=\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\int_0^\infty\frac{y^{s_1+s_2-2}}{e^y-1}dy\\&=\frac{\Gamma(1-s_1)}{\Gamma(s_2)}\sum_{n=1}^\infty\int_0^\infty y^{s_1+s_2-2}e^{-ny}dy\\&=\frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\Gamma(s_2)}\zeta(s_1+s_2-1)\end{aligned}$$

を得る．また, $\mathcal{C}$ をハンケル経路とすると (向きが逆なことに注意されたい),

$$\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_0^\infty \frac{y^{s_2-1}}{e^y-1}\int_0^\infty x^{s_1-1}h(x+y)dxdy = \frac{1}{\Gamma(s_1)\Gamma(s_2)(e^{2\pi is_1}-1)(e^{2\pi is_2}-1)}\int_{\mathcal{C}} \frac{y^{s_2-1}}{e^y-1}\int_{\mathcal{C}} x^{s_1-1}h(x+y)dxdy$$

を得る．この積分は $\Re s_1 < 1$ において収束する．ここで留数定理を用いると,

$$(17) = \frac{-2\pi i}{\Gamma(s_1)\Gamma(s_2)(e^{2\pi is_1}-1)(e^{2\pi is_2}-1)}\sum_{n\neq 0}\int_{\mathcal{C}} \frac{y^{s_2-1}}{e^y-1}(-y+2\pi in)^{s_1-1}dy$$

を得る．さらに経路を $0\to\infty$ に直すことで,

$$\begin{aligned}\int_{\mathcal{C}} \frac{y^{s_2-1}}{e^y-1}(-y+2\pi in)^{s_1-1}dy &= (e^{2\pi is_2}-1)\int_0^\infty \frac{y^{s_2-1}}{e^y-1}(-y+2\pi in)^{s_1-1}dy\\ &= \Gamma(s_2)(e^{2\pi is_2}-1)(2\pi n)^{s_1+s_2-1}e^{\pi i(s_1-s_2-1)/2}\sum_{0<m}\Psi(s_2,s_1+s_2;-2\pi inm)\end{aligned}$$

を得る．よって,

$$\begin{aligned}(17) &= (2\pi)^{s_1+s_2-1}\Gamma(1-s_1)e^{\pi i(1-s_1-s_2)/2}\sum_{n\neq 0,0<m}\Psi(s_2,s_1+s_2;-2\pi inm)n^{s_1+s_2-1}\\ &= (2\pi)^{s_1+s_2-1}\Gamma(1-s_1)\{e^{-\pi i(s_1+s_2-1)/2}F_-(s_1,s_2)+e^{\pi i(s_1+s_2-1)/2}F_+(s_1,s_2)\}\end{aligned}$$

Theorem 2.2.

$$\Psi(a,c;x) = x^{1-c}\Psi(a-c+1, 2-c; x)$$

証明　Mellin–Barnes 積分より,

$$\begin{aligned}
\Psi(a,c;x) &:= \frac{1}{\Gamma(a)}\int_0^\infty e^{-xt}t^{a-1}(1+t)^{b-a-1}dt \\
&= \frac{1}{2\pi i\Gamma(a)}\int_0^\infty t^{a-1}(1+t)^{b-a-1}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)(xt)^s dsdt \\
&= \frac{1}{2\pi i\Gamma(a)}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)x^s\frac{\Gamma(a+s)\Gamma(1-b-s)}{\Gamma(1+a-b)}ds \\
&= \frac{1}{\Gamma(a)}\sum_{0\le n}\left(\frac{\Gamma(a+n)\Gamma(1-b-n)}{n!}(-z)^n + z^{1-b}\frac{\Gamma(b-1-n)\Gamma(1+a-b+n)}{n!}(-z)^n\right)
\end{aligned}$$

となる. 最後の級数表示より定理を示せた. □

$$
\begin{aligned}
\frac{\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}}\zeta_{EZ,2}(s_1,s_2) &= \sum_{0<n,m}\frac{1}{n^{s_1}}\int_0^\infty x^{\frac{s_2}{2}-1}e^{-(n+m)^2\pi x}dx \\
&= \sum_{0<n}\frac{1}{n^{s_1}}\left(\int_0^1 x^{\frac{s_2}{2}-1}\sum_{0<m}e^{-(n+m)^2\pi x}dx+\int_1^\infty\sum_{0<m}x^{\frac{s_2}{2}-1}e^{-(n+m)^2\pi x}dx\right) \\
&= \sum_{0<n}\frac{1}{n^{s_1}}\left(\int_0^1 x^{\frac{s_2}{2}-1}\left\{\frac{1}{\sqrt{x}}\sum_{0<m}e^{-\frac{m^2\pi}{x}}+\frac{1}{2\sqrt{x}}-\frac{1}{2}-\sum_{m=1}^{n}e^{-m^2\pi x}\right\}dx\right. \\
&\qquad\left.+\int_1^\infty\left\{\sum_{0<m}x^{\frac{s_2}{2}-1}e^{-m^2\pi x}-\sum_{m=1}^{n}x^{\frac{s_2}{2}-1}e^{-m^2\pi x}\right\}dx\right) \\
&= -\frac{\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}}\zeta^\star_{EZ,2}(s_2,s_1)-\frac{1}{s_2(1-s_2)}\zeta_{EZ,1}(s_1) \\
&\qquad+\zeta_{EZ,1}(s_1)\int_1^\infty\left(x^{\frac{1-s_2}{2}-1}+x^{\frac{s_2}{2}-1}\right)\psi(x)dx
\end{aligned}
$$

よって,

$$\frac{\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}}\zeta_{EZ,2}(s_1,s_2)+\frac{\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}}\zeta_{EZ,2}(s_2,s_1)+\frac{\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}}\zeta_{EZ,1}(s_1+s_2)$$
$$=-\frac{1}{s_2(1-s_2)}\zeta_{EZ,1}(s_1)+\zeta_{EZ,1}(s_1)\int_1^\infty\left(x^{\frac{1-s_2}{2}-1}+x^{\frac{s_2}{2}-1}\right)\psi(x)dx$$

という表示を得る. また, 完備化された Riemann のゼータ関数に書き直すと,

$$\frac{\Gamma(\frac{s_2}{2})\pi^{-\frac{s_1}{2}}}{\pi^{\frac{s_2}{2}}\Gamma(\frac{s_1}{2})}\zeta_{EZ,2}(s_1,s_2)+\frac{\Gamma(\frac{s_2}{2})\pi^{-\frac{s_1}{2}}}{\pi^{\frac{s_2}{2}}\Gamma(\frac{s_1}{2})}\zeta_{EZ,2}(s_2,s_1)+\frac{\pi^{-\frac{s_1}{2}}\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}\Gamma(\frac{s_1}{2})}\zeta_{EZ,1}(s_1+s_2)$$
$$=-\frac{1}{s_2(1-s_2)}\xi(s_1)+\xi(s_1)\int_1^\infty\left(x^{\frac{1-s_2}{2}-1}+x^{\frac{s_2}{2}-1}\right)\psi(x)dx$$

を得る. したがって,

$$\frac{\Gamma(\frac{s_2}{2})\pi^{-\frac{s_1}{2}}}{\pi^{\frac{s_2}{2}}\Gamma(\frac{s_1}{2})}\zeta_{EZ,2}(s_1,s_2)+\frac{\Gamma(\frac{s_2}{2})\pi^{-\frac{s_1}{2}}}{\pi^{\frac{s_2}{2}}\Gamma(\frac{s_1}{2})}\zeta_{EZ,2}(s_2,s_1)+\frac{\pi^{-\frac{s_1}{2}}\Gamma(\frac{s_2}{2})}{\pi^{\frac{s_2}{2}}\Gamma(\frac{s_1}{2})}\zeta_{EZ,1}(s_1+s_2)$$
$$=\frac{\Gamma(\frac{1-s_2}{2})\pi^{-\frac{1-s_1}{2}}}{\pi^{\frac{1-s_2}{2}}\Gamma(\frac{1-s_1}{2})}\zeta_{EZ,2}(1-s_1,1-s_2)+\frac{\Gamma(\frac{1-s_2}{2})\pi^{-\frac{1-s_1}{2}}}{\pi^{\frac{1-s_2}{2}}\Gamma(\frac{1-s_1}{2})}\zeta_{EZ,2}(1-s_2,1-s_1)$$
$$+\frac{\pi^{-\frac{1-s_1}{2}}\Gamma(\frac{1-s_2}{2})}{\pi^{\frac{1-s_2}{2}}\Gamma(\frac{1-s_1}{2})}\zeta_{EZ,1}(2-s_1-s_2)$$

となる. これは $\zeta_{EZ,1}(s_1)\zeta_{EZ,1}(s_2)=\zeta_{EZ,2}(s_1,s_2)+\zeta_{EZ,2}(s_2,s_1)+\zeta_{EZ,1}(s_1+s_2)$ より自明である

Definition 1.4 (Multiple confluent hypergeometric functions)**.** Let a be a positive integer. We define the multiple confluent hypergeometric functions by the following infinite integral

$$\begin{aligned}
&\Psi_a(h_1,\ldots,h_{a+1};x_1,\ldots,x_a;\delta)\\
&:=\frac{1}{\Gamma(h_2)\cdots\Gamma(h_{a+1})}\int_0^{\infty e^{i\phi}} e^{-x_a t_a} t_a^{h_{a+1}-1}\int_0^{\infty e^{i\phi}} e^{-x_{a-1}t_{a-1}} t_{a-1}^{h_a-1}\\
&\qquad\times\int_0^{\infty e^{i\phi}}\cdots\int_0^{\infty e^{i\phi}} e^{-x_2t_2}t_2^{h_3-1}\int_0^{\infty e^{i\phi}} e^{-x_1t_1}t_1^{h_2-1}(\delta+t_1+t_2+\cdots+t_a)^{h_1-1}dt_1\cdots dt_a,
\end{aligned}$$

where $0\le\delta\le 1$, complex variables $h_2,\ldots,h_{a+1}$ satisfy $\Re h_k>0$ for $2\le k\le a+1$ and ϕ satisfies $|\phi+\arg x|<\pi/2$.

We introduce two functions

$$\begin{aligned}
\mathscr{F}_{\pm}^{r}(s_1,\ldots,s_r):=&\sum_{k_1,\ldots,k_{r-1}=1}^{\infty}\sigma_{s_1+\cdots+s_r-1}(k_1,\ldots,k_{r-1})\\
&\times\Psi_{r-1}(s_1,\ldots,s_r;\pm 2\pi i k_1,\pm 2\pi i(k_1+k_2),\ldots,\pm 2\pi i(k_1+\cdots+k_{r-1});1).
\end{aligned}$$

which performs the same role as (1.2) and (1.5). This function is absolutely convergent when $\Re s_1<0$ and $\Re s_k>1$ for $2\le k\le r$. (See Theorem 3.2.) Moreover it can be continued meromorphically to $\mathfrak{A}_r$ space. (See Theorem 4.2.)

Lemma 2.1. Let n be a positive integer. We have

$$\begin{aligned}&\int_0^\infty x^{h_1-1}(1+x)^{h_2-1}(1+\alpha_1 x)^{h_3-1}\cdots(1+\alpha_n x)^{h_{n+2}-1}dx\\ &=\frac{\Gamma(h_1)\Gamma(1-h_1-h_2-\cdots-h_{n+2}+n)}{\Gamma(1-h_2-h_3-\cdots-h_{n+2}+n)}\\ &\qquad\times F_D^{(n)}(1-h_3,\dots,1-h_{n+2};h_1,1-h_2-h_3-\cdots-h_{n+2}+n;1-\alpha_1,\dots,1-\alpha_n),\end{aligned}$$

where h_j, α_k are complex variables for $1\le j\le n+2$ and $1\le k\le n$.

Proof. We prove it by the change of variables $x=\frac{t}{1-t}$ as follows

$$\begin{aligned}&\int_0^\infty x^{h_1-1}(1+x)^{h_2-1}(1+\alpha_1 x)^{h_3-1}\cdots(1+\alpha_n x)^{h_{n+2}-1}dx\\ &=\int_0^1\left(\frac{t}{1-t}\right)^{h_1-1}\left(\frac{1}{1-t}\right)^{h_2-1}\left(1+\alpha_1\frac{t}{1-t}\right)^{h_3-1}\cdots\left(1+\alpha_n\frac{t}{1-t}\right)^{h_{n+2}-1}\frac{dt}{(1-t)^2}\\ &=\int_0^1 t^{h_1-1}(1-t)^{n-h_1-\cdots-h_{n+2}}(1-(1-\alpha_1)t)^{h_3-1}\cdots(1-(1-\alpha_n)t)^{h_{n+2}-1}dt\\ &=\sum_{m_1,\dots,m_n=0}^\infty\frac{(1-h_3)_{m_1}(1-\alpha_1)^{m_1}}{m_1!}\cdots\frac{(1-h_{n+2})_{m_n}(1-\alpha_n)^{m_n}}{m_n!}\\ &\qquad\qquad\times\int_0^1 t^{h_1+m_1+\cdots+m_n-1}(1-t)^{n-h_1-\cdots-h_{n+2}}dt\\ &=\sum_{m_1,\dots,m_n=0}^\infty\frac{(1-h_3)_{m_1}(1-\alpha_1)^{m_1}}{m_1!}\cdots\frac{(1-h_{n+2})_{m_n}(1-\alpha_n)^{m_n}}{m_n!}\\ &\qquad\times\frac{\Gamma(h_1+m_1+\cdots+m_n)\Gamma(1-h_1-h_2-\cdots-h_{n+2}+n)}{\Gamma(1-h_2-h_3-\cdots-h_{n+2}+m_1+\cdots+m_n+n)}.\end{aligned}$$

We used the formula $\int_0^1 t^{n-1}(1-t)^{m-1}dt=\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$ in the last equality. $\square$

Lemma 2.2. Let a be a positive integer and let complex variables $h_1, \ldots, h_{a+1}$ satisfy $h_1 < 1$ and $0 < h_2 + \cdots + h_{a+1}$. We have

$$\Psi_a(h_1, \ldots, h_{a+1}; x_1, \ldots, x_a; 1)$$

$$= x_1^{h_3+\cdots+h_{a+1}} x_2^{-h_3} \cdots x_a^{-h_{a+1}} \sum_{m_1=0}^{\infty} \frac{(h_3)_{m_1}(1-\frac{x_1}{x_2})^{m_1}}{m_1!} \cdots \sum_{m_{a-1}=0}^{\infty} \frac{(h_{a+1})_{m_{a-1}}(1-\frac{x_1}{x_a})^{m_{a-1}}}{m_{a-1}!} \tag{2.1}$$

$$\times (1-h_1)_{m_1+\cdots+m_{a-1}} \Psi(h_2 + \cdots + h_{a+1} + m_1 + \cdots + m_{a-1}, h_1 + \cdots + h_{a+1}; x_1).$$

Proof. By making the substitution $t_1 = (\delta + t_2 + \cdots + t_a)u$, the integral becomes

$$\begin{aligned}
&\Psi_a(h_1, \ldots, h_{a+1}; x_1, \ldots, x_a; \delta) \\
&= \frac{1}{\Gamma(h_2)\cdots\Gamma(h_{a+1})} \int_0^\infty e^{-x_a t_a} t_a^{h_{a+1}-1} \int_0^\infty e^{-x_{a-1}t_{a-1}} t_{a-1}^{h_a-1} \cdots \int_0^\infty e^{-x_2 t_2} t_2^{h_3-1} \\
&\qquad \times \int_0^\infty e^{-x_1 u(\delta+t_2+\cdots+t_a)} (\delta + t_2 + \cdots + t_a)^{h_1+h_2-1} u^{h_2-1} (1+u)^{h_1-1} du dt_2 \cdots dt_a \\
&= \frac{1}{\Gamma(h_3)\cdots\Gamma(h_{a+1})} \\
&\quad \times \int_0^\infty e^{-x_a t_a} t_a^{h_{a+1}-1} \int_0^\infty e^{-x_{a-1}t_{a-1}} t_{a-1}^{h_a-1} \cdots \int_0^\infty e^{-x_2 t_2} t_2^{h_3-1} (\delta + t_2 + \cdots + t_a)^{h_1+h_2-1} \\
&\qquad\qquad \times \Psi(h_2, h_1 + h_2; x_1(\delta + t_2 + \cdots + t_a)) dt_2 \cdots dt_a.
\end{aligned} \tag{2.2}$$

Recalling the well-known and beautiful property of $\Psi(b, c; x)$:

$$\Psi(b, c; x) = x^{1-c}\Psi(b - c + 1, 2 - c; x) \tag{2.3}$$

shown in [5, 6.5 (6)], we can show

$$\begin{aligned}
(2.2) &= \frac{x_1^{1-h_1-h_2}}{\Gamma(1-h_1)\Gamma(h_3)\cdots\Gamma(h_{a+1})} \int_0^\infty e^{-(x_a+x_1t_1)t_a} t_a^{h_{a+1}-1} \int_0^\infty e^{-(x_{a-1}+x_1t_1)t_{a-1}} t_{a-1}^{h_a-1} \\
&\qquad \times \int_0^\infty \cdots \int_0^\infty e^{-(x_2+x_1t_1)t_2} t_2^{h_3-1} \int_0^\infty e^{-\delta x_1 t_1} t_1^{-h_1} (1+t_1)^{-h_2} dt_1 \cdots dt_a \\
&= \frac{x_1^{1-h_1-h_2}}{\Gamma(1-h_1)} \int_0^\infty e^{-\delta x_1 t_1} t_1^{-h_1} (1+t_1)^{-h_2} (x_2 + x_1 t_1)^{-h_3} \cdots (x_a + x_1 t_1)^{-h_{a+1}} dt_1.
\end{aligned} \tag{2.4}$$

Putting $\delta = 1$, we see that the right-hand side of (2.4) is

$$= \frac{x_1^{1-h_1-h_2} x_2^{-h_3} \cdots x_a^{-h_{a+1}}}{2\pi i \Gamma(1-h_1)} \int_{\mathcal{M}} \Gamma(-s) x_1^s \tag{2.5}$$
$$\times \int_0^\infty t_1^{s-h_1} (1+t_1)^{-h_2} \left(1 + \left(\frac{x_1}{x_2}\right) t_1\right)^{-h_3} \cdots \left(1 + \left(\frac{x_1}{x_a}\right) t_1\right)^{-h_{a+1}} dt_1 ds.$$

Here we used the formula $e^{-z} = \frac{1}{2\pi i} \int_{\mathcal{M}} \Gamma(-s) z^s ds$. We define the integration contour $\mathcal{M}$ as the vertical line running from $c - i\infty$ to $c + i\infty$. In order to choose c so that all singularities of $\Gamma(-s)$ and $\Gamma(h_1 + \cdots + h_{a+1} - s - 1)$ lie to the right of the contour, we assume $\Re(h_1 - 1) < c < 0$. Applying Lemma 2.1 and summing over the poles of $\Gamma(-s)$ and $\Gamma(h_1 + \cdots + h_{a+1} - s - 1)$, we have

$$(2.5) = \frac{x_1^{1-h_1-h_2} x_2^{-h_3} \cdots x_a^{-h_{a+1}}}{2\pi i \Gamma(1-h_1)} \sum_{m_1, \ldots, m_{a-1}=0}^{\infty} \frac{(h_3)_{m_1} (1 - \frac{x_1}{x_2})^{m_1}}{m_1!} \cdots \frac{(h_{a+1})_{m_{a-1}} (1 - \frac{x_1}{x_a})^{m_{a-1}}}{m_{a-1}!}$$
$$\times \int_{\mathcal{M}} \Gamma(-s) x_1^s \frac{\Gamma(s - h_1 + m_1 + \cdots + m_{a-1} + 1) \Gamma(h_1 + \cdots + h_{a+1} - s - 1)}{\Gamma(h_2 + h_3 + \cdots + h_{a+1} + m_1 + \cdots + m_{a-1})} ds$$

$$
\begin{aligned}
&= \frac{x_1^{1-h_1-h_2} x_2^{-h_3} \cdots x_a^{-h_{a+1}}}{\Gamma(1-h_1)} \sum_{m_1,\dots,m_{a-1}=0}^{\infty} \frac{(h_3)_{m_1}(1-\frac{x_1}{x_2})^{m_1}}{m_1!} \cdots \frac{(h_{a+1})_{m_{a-1}}(1-\frac{x_1}{x_a})^{m_{a-1}}}{m_{a-1}!} \\
&\quad \times \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} \Bigg(x_1^\ell \frac{\Gamma(\ell-h_1+m_1+\cdots+m_{a-1}+1)\Gamma(h_1+\cdots+h_{a+1}-\ell-1)}{\Gamma(h_2+h_3+\cdots+h_{a+1}+m_1+\cdots+m_{a-1})} \\
&\quad + x_1^{h_1+\cdots+h_{a+1}-1+\ell} \frac{\Gamma(1-h_1-\cdots-h_{a+1}-\ell)\Gamma(h_2+h_3+\cdots+h_{a+1}+\ell+m_1+\cdots+m_{a-1})}{\Gamma(h_2+h_3+\cdots+h_{a+1}+m_1+\cdots+m_{a-1})} \Bigg).
\end{aligned} \tag{2.6}
$$

By using the formula $\frac{1}{(1-x)_n} = (-1)^n (x)_{-n}$, we can show

$$
\begin{aligned}
(2.6) &= \frac{x_1^{1-h_1-h_2} x_2^{-h_3} \cdots x_a^{-h_{a+1}}}{\Gamma(1-h_1)} \sum_{m_1,\dots,m_{a-1}=0}^{\infty} \frac{(h_3)_{m_1}(1-\frac{x_1}{x_2})^{m_1}}{m_1!} \cdots \frac{(h_{a+1})_{m_{a-1}}(1-\frac{x_1}{x_a})^{m_{a-1}}}{m_{a-1}!} \\
&\quad \times \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \Bigg(x_1^\ell \frac{\Gamma(1-h_1+m_1+\cdots+m_{a-1})\Gamma(h_1+\cdots+h_{a+1}-1)(1-h_1+m_1+\cdots+m_{a-1})_\ell}{(2-h_1-\cdots-h_{a+1})_\ell \Gamma(h_2+h_3+\cdots+h_{a+1}+m_1+\cdots+m_{a-1})} \\
&\quad + x_1^{h_1+\cdots+h_{a+1}-1+\ell} \frac{\Gamma(1-h_1-\cdots-h_{a+1})(h_2+h_3+\cdots+h_{a+1}+m_1+\cdots+m_{a-1})_\ell}{(h_1+\cdots+h_{a+1})_\ell} \Bigg) \\
&= \frac{x_1^{h_3+\cdots+h_{a+1}} x_2^{-h_3} \cdots x_a^{-h_{a+1}}}{\Gamma(1-h_1)} \sum_{m_1,\dots,m_{a-1}=0}^{\infty} \frac{(h_3)_{m_1}(1-\frac{x_1}{x_2})^{m_1}}{m_1!} \cdots \frac{(h_{a+1})_{m_{a-1}}(1-\frac{x_1}{x_a})^{m_{a-1}}}{m_{a-1}!} \\
&\qquad \times \Gamma(1-h_1+m_1+\cdots+m_{a-1}) \\
&\quad \times \Bigg(x_1^{1-h_1-\cdots-h_{a+1}} \frac{\Gamma(h_1+\cdots+h_{a+1}-1)}{\Gamma(h_2+\cdots+h_{a+1}+m_1+\cdots+m_{a-1})} {}_1F_1 \left[\begin{matrix} 1-h_1+m_1+\cdots+m_{a-1} \\ 2-h_1-\cdots-h_{a+1} \end{matrix} ; x_1 \right] \\
&\quad + \frac{\Gamma(1-h_1-\cdots-h_{a+1})}{\Gamma(1-h_1+m_1+\cdots+m_{a-1})} {}_1F_1 \left[\begin{matrix} h_2+\cdots+h_{a+1}+m_1+\cdots+m_{a-1} \\ h_1+\cdots+h_{a+1} \end{matrix} ; x_1 \right] \Bigg).
\end{aligned} \tag{2.7}
$$

Proof. By substituting $t_j = 2\pi i n\eta_j$, we can show

$$
\begin{aligned}
&\mathscr{G}_r(s_1,\ldots,s_r)\\
&= \frac{(2\pi i)^{s_1+\cdots+s_r-1}\Gamma(1-s_1)}{\Gamma(s_2)\cdots\Gamma(s_r)} \sum_{n=1}^{\infty} n^{s_1+\cdots+s_r-1} \sum_{m_1,\ldots,m_{r-1}=1}^{\infty} \int_0^{i\infty} e^{-2\pi i n m_{r-1}\eta_r} \eta_r^{s_r-1}\\
&\qquad \times \int_0^{i\infty}\cdots\int_0^{i\infty} e^{-2\pi i n m_2(\eta_3+\cdots+\eta_r)}\eta_3^{s_3-1}\int_0^{i\infty} e^{-2\pi i n m_1(\eta_2+\cdots+\eta_r)}\eta_2^{s_2-1}(1+\eta_2+\eta_3+\cdots+\eta_r)^{s_1-1}d\eta_2\cdots d\eta_r\\
&\qquad\qquad + \frac{(-2\pi i)^{s_1+\cdots+s_r-1}\Gamma(1-s_1)}{\Gamma(s_2)\cdots\Gamma(s_r)} \sum_{n=1}^{\infty} n^{s_1+\cdots+s_r-1} \sum_{m_1,\ldots,m_{r-1}=1}^{\infty} \int_0^{-i\infty} e^{2\pi i n m_{r-1}\eta_r} \eta_r^{s_r-1}\\
&\qquad \times \int_0^{-i\infty}\cdots\int_0^{-i\infty} e^{2\pi i n m_2(\eta_3+\cdots+\eta_r)}\eta_3^{s_3-1}\int_0^{-i\infty} e^{2\pi i n m_1(\eta_2+\cdots+\eta_r)}\eta_2^{s_2-1}(1+\eta_2+\eta_3+\cdots+\eta_r)^{s_1-1}d\eta_2\cdots d\eta_r\\
&= \frac{(2\pi i)^{s_1+\cdots+s_r-1}\Gamma(1-s_1)}{\Gamma(s_2)\cdots\Gamma(s_r)} \sum_{n=1}^{\infty} n^{s_1+\cdots+s_r-1} \sum_{m_1,\ldots,m_{r-1}=1}^{\infty} \int_0^{i\infty} e^{-2\pi i n(m_1+\cdots+m_{r-1})\eta_r} \eta_r^{s_r-1}\\
&\qquad \times \int_0^{i\infty}\cdots\int_0^{i\infty} e^{-2\pi i n(m_1+m_2)\eta_3}\eta_3^{s_3-1}\int_0^{i\infty} e^{-2\pi i n m_1\eta_2}\eta_2^{s_2-1}(1+\eta_2+\eta_3+\cdots+\eta_r)^{s_1-1}d\eta_2\cdots d\eta_r\\
&\qquad + \frac{(-2\pi i)^{s_1+\cdots+s_r-1}\Gamma(1-s_1)}{\Gamma(s_2)\cdots\Gamma(s_r)} \sum_{n=1}^{\infty} n^{s_1+\cdots+s_r-1} \sum_{m_1,\ldots,m_{r-1}=1}^{\infty} \int_0^{-i\infty} e^{2\pi i n(m_1+\cdots+m_{r-1})\eta_r} \eta_r^{s_r-1}\\
&\qquad \times \int_0^{-i\infty}\cdots\int_0^{-i\infty} e^{2\pi i n(m_1+m_2)\eta_3}\eta_3^{s_3-1}\int_0^{-i\infty} e^{2\pi i n m_1\eta_2}\eta_2^{s_2-1}(1+\eta_2+\eta_3+\cdots+\eta_r)^{s_1-1}d\eta_2\cdots d\eta_r\\
&= (2\pi)^{s_1+\cdots+s_r-1}e^{\frac{\pi i(s_1+\cdots+s_r-1)}{2}}\Gamma(1-s_1)\mathscr{F}_+^r(s_1,\ldots,s_r) + (2\pi)^{s_1+\cdots+s_r-1}e^{\frac{\pi i(1-s_1-\cdots-s_r)}{2}}\Gamma(1-s_1)\mathscr{F}_-^r(s_1,\ldots,s_r).
\end{aligned}
$$

This complete the proof. $\square$

Theorem 1.6. Let r be a positive integer satisfying $r \geq 2$. When the complex variables $(s_1, \dots, s_r)$ are contained in $\mathfrak{A}_r$, we have

$$\frac{\mathscr{G}_r(1 - \mathrm{wt}(\boldsymbol{s}) + s_1, 1 - \mathrm{wt}(\boldsymbol{s}) + s_2, s_3, \dots, s_r)}{\Gamma(\mathrm{wt}(\boldsymbol{s}) - s_1) i^{\mathrm{wt}(\boldsymbol{s})-1}} + e^{\frac{\pi i}{2}(\mathrm{wt}(\boldsymbol{s})-1)} \mathscr{F}_+^r(s_1, \dots, s_r) + e^{-\frac{\pi i}{2}(\mathrm{wt}(\boldsymbol{s})-1)} \mathscr{F}_-^r(s_1, \dots, s_r)$$

$$= \frac{\mathscr{G}_r(s_1, \dots, s_r)}{\Gamma(1 - s_1)(2\pi)^{\mathrm{wt}(\boldsymbol{s})-1}}$$

$$+ e^{-\frac{\pi i}{2}(\mathrm{wt}(\boldsymbol{s})-1)} \sum_{k_1, \dots, k_{r-1}=1}^{\infty} \sigma_{EZ, r-1}(2\mathrm{wt}(\boldsymbol{s}) - s_1 - s_2 - 1, -s_3, \dots, -s_r; k_1, \dots, k_r)$$

$$\times \sum_{m_1, \dots, m_{r-2}=0}^{\infty} \frac{(s_3)_{m_1}(1 - \frac{k_1}{k_1+k_2})^{m_1}}{m_1!} \cdots \frac{(s_r)_{m_{r-2}}(1 - \frac{k_1}{k_1+\cdots+k_{r-1}})^{m_{r-2}}}{m_{r-2}!}$$

$$\times (\mathrm{wt}(\boldsymbol{s}) - s_1)_{m_1+\cdots+m_{r-2}} \{\Psi(\mathrm{wt}(\boldsymbol{s}) - s_1 + m_1 + \cdots + m_{r-2}, \mathrm{wt}(\boldsymbol{s}); 2\pi i k_1; 1)$$

$$+ \Psi(\mathrm{wt}(\boldsymbol{s}) - s_1 + m_1 + \cdots + m_{r-2}, \mathrm{wt}(\boldsymbol{s}); -2\pi i k_1; 1)\}.$$

Here we set $\boldsymbol{s} \coloneqq \{s_1, \dots, s_r\}$ and define $\mathrm{wt}(\boldsymbol{s}) = s_1 + \cdots + s_r$. When all components are positive integers, $\boldsymbol{s}$ is called an index, and $\mathrm{wt}(\boldsymbol{s})$ is referred to as its weight.

1.1. **Examples.**

The following equation evidently holds by definition of the multiple confluent hypergeometric functions,

$$\Psi_1(h_1, h_2; x_1; 1) = \Psi(h_2, h_1 + h_2; x_1).$$

Hence, it is straightforward to verify that Theorem 1.2 holds by the Theorem 1.6 in the case when $r = 2$. Here we understand that $m_1, \ldots, m_{r-2} = 0$ if $r = 2$.

In the case $r = 3$ where we have

$$\begin{aligned}
&\frac{\mathscr{G}_3(1 - s_2 - s_3, 1 - s_1 - s_3, s_3)}{\Gamma(s_2 + s_3) i^{s_1+s_2+s_3-1}} + e^{\frac{\pi}{2}(s_1+s_2+s_3-1)} \mathscr{F}_+^r(s_1, s_2, s_3) + e^{-\frac{\pi}{2}(s_1+s_2+s_3-1)} \mathscr{F}_-^r(s_1, s_2, s_3) \\
&= \frac{\mathscr{G}_3(s_1, s_2, s_3)}{\Gamma(1 - s_1)(2\pi)^{s_1+s_2+s_3-1}} + \sum_{k_1, k_2=1}^{\infty} \sigma_{EZ,2}(s_1 + s_2 + 2s_3 - 1, -s_3; k_1, k_2) \\
&\quad \times \sum_{m_1=0}^{\infty} \frac{(s_3)_{m_1} (1 - \frac{k_1}{k_1+k_2})^{m_1}}{m_1!} (s_2 + s_3)_{m_1} \{\Psi(s_2 + s_3 + m_1, s_1 + s_2 + s_3; 2\pi i k_1; 1) \\
&\qquad + \Psi(s_2 + s_3 + m_1, s_1 + s_2 + s_3; -2\pi i k_1; 1)\}.
\end{aligned}$$

Thank you for listening