# モノイダル圏への関手と(余)代数の対応について

## Thm

$\mathcal{C}$ : category , $\mathcal{D}$ : locally small monoidal category , $F : \mathcal{C} \longrightarrow \mathcal{D}$ : functor

$Nat(-\otimes F, F) : \mathcal{D}^{op} \longrightarrow Set$ に対して、$A_F \in \mathcal{D}$ と 自然同型 $\theta : Hom_{\mathcal{D}}(-, A_F) \longrightarrow Nat(-\otimes F, F)$

が存在するとき、以下が成立する。ただし、$\varphi = \theta_{A_F}(id_{A_F}) \in Nat(A_F \otimes F, F)$ とする。

(i) $\forall M \in \mathcal{D}$ , $\forall \alpha \in Nat(M\otimes F, F)$ に対して、$\theta_M^{-1}(\alpha)$ は 以下の図式を可換にする unique な射である。

$$\begin{tikzcd}
A_F \otimes F \arrow[r, "\varphi"] & F \\
M \otimes F \arrow[u, "\theta_M^{-1}(\alpha)\otimes id_F"] \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

(ii) $A_F$ は $\mathcal{D}$ における monoid str を持つ。

(iii) $U : {}_{A_F}M \longrightarrow \mathcal{D}$ を forgetful functor とすると、functor $G : \mathcal{C} \longrightarrow {}_{A_F}M$ が存在し、以下の図式は可換となる。

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "G"] \arrow[dr, "F"'] & \circlearrowleft & {}_{A_F}M \arrow[dl, "U"] \\
& \mathcal{D} &
\end{tikzcd}$$

(proof)

$\mu_{A_F} = \theta_{A_F\otimes A_F}^{-1}(\varphi \circ (id_{A_F} \otimes \varphi) \circ a_{A_F, A_F, F})$ とおく。

$\mu_{A_F} : A_F \otimes A_F \longrightarrow A_F$ は $\mathcal{D}$ の morphism より $\theta$ の naturality より

$$\begin{tikzcd}
Hom_{\mathcal{D}}(A_F, A_F) \arrow[r, "\mu_{A_F}^*"] \arrow[d, "\theta_{A_F}"'] & Hom_{\mathcal{D}}(A_F\otimes A_F, A_F) \arrow[d, "\theta_{A_F\otimes A_F}"] \\
Nat(A_F\otimes F, F) \arrow[r, "\mu_{A_F}^*"'] & Nat((A_F\otimes A_F)\otimes F, A_F)
\end{tikzcd}$$

$$\begin{tikzcd}
id_{A_F} \arrow[r, mapsto] \arrow[d, mapsto] & \mu_{A_F} \arrow[d, mapsto] \\
\varphi \arrow[r, mapsto] \arrow[ur, phantom, "\circlearrowleft"] & \varphi\circ(id_{A_F}\otimes\varphi)\circ a_{A_F,A_F,F}
\end{tikzcd}$$

$\therefore\ \varphi \circ (\mu_{A_F} \otimes id_F) = \varphi \circ (id_{A_F} \otimes \varphi) \circ a_{A_F, A_F, F}$

$$\begin{tikzcd}
(A_F\otimes A_F)\otimes F \arrow[rr, "a_{A_F,A_F,F}"] \arrow[d, "\mu_{A_F}\otimes id_F"'] & & A_F\otimes(A_F\otimes F) \arrow[d, "id_{A_F}\otimes\varphi"] \\
A_F\otimes F \arrow[dr, "\varphi"'] & \circlearrowleft & A_F\otimes F \arrow[dl, "\varphi"] \\
& F &
\end{tikzcd}$$

(i) $\forall M \in \mathcal{D}$ , $\forall \alpha \in Nat(M\otimes F, F)$ をとる。 $\forall f : M \longrightarrow A_F$ in $\mathcal{D}$ に対し、$\theta$ : natural iso より

$$\begin{tikzcd}
Hom_{\mathcal{D}}(A_F, A_F) \arrow[r, "f^*"] \arrow[d, "\theta_{A_F}"'] & Hom_{\mathcal{D}}(M, A_F) \arrow[d, "\theta_M"] \\
Nat(A_F\otimes F, F) \arrow[r, "(f\otimes id_F)^*"'] & Nat(M\otimes F, F)
\end{tikzcd}$$

$$\begin{tikzcd}
id_{A_F} \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] \\
\varphi \arrow[r, mapsto] \arrow[ur, phantom, "\circlearrowleft"] & \theta_M(f)
\end{tikzcd}$$

$\therefore\ \varphi \circ (f \otimes id_F) = \theta_M(f)$

特に $f = \theta_M^{-1}(\alpha)$ のとき $\varphi \circ (\theta_M^{-1}(\alpha) \otimes id_F) = \alpha$

また、$\varphi \circ (f \otimes id_F) = \alpha$ のとき、$\theta_M(f) = \alpha$ より

$f = \theta_M^{-1}(\alpha)$ となり、射の uniqueness が示せる。

(ii) 以下の図式の (*) の五角形は可換となるので、$\mu_{A_F}$ は結合律をみたす。

$$\begin{tikzcd}[column sep=small]
 & & & A_F\otimes(A_F\otimes F) \arrow[dddddlll, bend right=40, "id_{A_F}\otimes\varphi"'] \arrow[dddddrrr, bend left=40, "id_{A_F}\otimes\varphi"] & & & \\
 & (A_F\otimes A_F)\otimes F \arrow[urr] \arrow[ddddl, bend right, "\mu_{A_F}\otimes id_F"'] & & A_F\otimes(A_F\otimes(A_F\otimes F)) \arrow[u, "id_{A_F}\otimes(id_{A_F}\otimes\varphi)"'] & & & \\
 & (A_F\otimes A_F)\otimes(A_F\otimes F) \arrow[u, "(id_{A_F}\otimes id_{A_F})\otimes\varphi"] \arrow[urr] \arrow[dl, "\mu_{A_F}\otimes(id_{A_F}\otimes id_F)"'] & & & & A_F\otimes((A_F\otimes A_F)\otimes F) \arrow[ull] \arrow[dr, "id_{A_F}\otimes(\mu_{A_F}\otimes id_F)"] & \\
A_F\otimes(A_F\otimes F) \arrow[dd, "id_{A_F}\otimes\varphi"'] & & ((A_F\otimes A_F)\otimes A_F)\otimes F \arrow[ul] \arrow[rr] \arrow[dl, "(\mu_{A_F}\otimes id_{A_F})\otimes id_F"'] & & (A_F\otimes(A_F\otimes A_F))\otimes F \arrow[ur] \arrow[dr, "(id_{A_F}\otimes\mu_{A_F})\otimes id_F"] & & A_F\otimes(A_F\otimes F) \arrow[dd, "id_{A_F}\otimes\varphi"] \\
 & (A_F\otimes A_F)\otimes F \arrow[ul] \arrow[drr, "\mu_{A_F}\otimes id_F"] & & (*) & & (A_F\otimes A_F)\otimes F \arrow[ur] \arrow[dll, "\mu_{A_F}\otimes id_F"'] & \\
A_F\otimes F \arrow[drrr, "\varphi"'] & & & A_F\otimes F \arrow[d, "\varphi"] & & & A_F\otimes F \arrow[dlll, "\varphi"] \\
 & & & F & & &
\end{tikzcd}$$

$l_F : I\otimes F \longrightarrow F$ とすると $l_F \in Nat(I\otimes F, F)$ $\eta_{A_F} := \theta_I^{-1}(l_F) \in Hom_{\mathcal{D}}(I, A_F)$ とおく

$\theta$ の naturality より、

$$\begin{tikzcd}
Hom_{\mathcal{D}}(A_F, A_F) \arrow[r, "\eta_{A_F}^*"] \arrow[d, "\theta_{A_F}"'] & Hom_{\mathcal{D}}(I, A_F) \arrow[d, "\theta_I"] \\
Nat(A_F\otimes F, F) \arrow[r, "(\eta_{A_F}\otimes id_F)^*"'] & Nat(I\otimes F, F)
\end{tikzcd}$$

$$\begin{tikzcd}
id_{A_F} \arrow[r, mapsto] \arrow[d, mapsto] & \eta_{A_F} \arrow[d, mapsto] \\
\varphi \arrow[r, mapsto] & l_F
\end{tikzcd}$$

$\therefore\ \varphi\circ(\eta_{A_F}\otimes id_F) = l_{F(-)}$

$$\begin{tikzcd}
(I\otimes A_F)\otimes F \arrow[rrr, bend left, "(\eta_{A_F}\otimes id_{A_F})\otimes id_F"] \arrow[d] \arrow[dd, bend right=50, "l_{A_F}\otimes id_F"'] & & A_F\otimes(A_F\otimes F) \arrow[d, "id_{A_F}\otimes\varphi"] & (A_F\otimes A_F)\otimes F \arrow[l] \arrow[d, "\mu_{A_F}\otimes id_F"] \\
I\otimes(A_F\otimes F) \arrow[urr, "\eta_{A_F}\otimes(id_{A_F}\otimes id_F)"] \arrow[r, "id_I\otimes\varphi"] \arrow[d, "l_{A_F\otimes F}"'] & I\otimes F \arrow[r, "\eta_{A_F}\otimes id_F"] \arrow[d, "l_F"] & A_F\otimes F \arrow[dl, "\varphi"] & A_F\otimes F \arrow[dll, "\varphi"] \\
A_F\otimes F \arrow[r, "\varphi"'] & F & &
\end{tikzcd}$$

$\therefore$
$$\begin{tikzcd}
I\otimes A_F \arrow[r, "\eta_{A_F}\otimes id_{A_F}"] \arrow[dr, "l_{A_F}"'] & A_F\otimes A_F \arrow[d, "\mu_{A_F}"] \\
 & A_F
\end{tikzcd}
\qquad
\begin{tikzcd}
A_F\otimes A_F \arrow[d, "\mu_{A_F}"'] & A_F\otimes I \arrow[l, "id_{A_F}\otimes\eta_{A_F}"'] \arrow[dl, "r_{A_F}"] \\
A_F &
\end{tikzcd}$$

も同様に示すことができる。

(iii) $\forall X\in\mathcal{C}$ に対し、$\varphi_X : A_F\otimes F(X) \longrightarrow F(X)$ in $\mathcal{D}$ が定まる。

$$\begin{tikzcd}
(A_F\otimes A_F)\otimes F \arrow[r, "a_{A_F,A_F,F}"] \arrow[d, "\mu_{A_F}\otimes id_F"'] & A_F\otimes(A_F\otimes F) \arrow[d, "id_{A_F}\otimes\varphi"] \\
A_F\otimes F \arrow[dr, "\varphi"'] & A_F\otimes F \arrow[d, "\varphi"] \\
 & F
\end{tikzcd}
\quad \text{より、} \quad
\begin{tikzcd}
(A_F\otimes A_F)\otimes F(X) \arrow[r, "a_{A_F,A_F,F(X)}"] \arrow[d, "\mu_{A_F}\otimes id_F"'] & A_F\otimes(A_F\otimes F(X)) \arrow[d, "id_{A_F}\otimes\varphi_X"] \\
A_F\otimes F(X) \arrow[dr, "\varphi_X"'] & A_F\otimes F(X) \arrow[d, "\varphi_X"] \\
 & F(X)
\end{tikzcd}$$

$\varphi\circ(\eta_{A_F}\otimes id_F) = l_{F(-)}$ より $\varphi_X\circ(\eta_{A_F}\otimes id_{F(X)}) = l_{F(X)}$

$$\begin{tikzcd}
I\otimes F(X) \arrow[r, "\eta_{A_F}\otimes id_{F(X)}"] \arrow[dr, "l_{F(X)}"'] & A_F\otimes F(X) \arrow[d, "\varphi_X"] \\
 & F(X)
\end{tikzcd}$$

$\therefore\ (F(X), \varphi_X) \in {}_{A_F}\mathcal{M}$

また $f: X \longrightarrow Y$ in $\mathcal{C}$ に対して, $\varphi: A_F \otimes F \longrightarrow F$ の naturality より

$$\begin{tikzcd}
A_F \otimes F(X) \arrow[r, "id_{A_F} \otimes F(f)"] \arrow[d, "\varphi_X"'] \arrow[dr, phantom, "\circlearrowright"] & A_F \otimes F(Y) \arrow[d, "\varphi_Y"] \\
F(X) \arrow[r, "F(f)"'] & F(Y)
\end{tikzcd}$$

$\therefore$ $F(f): F(X) \longrightarrow F(Y)$ は ${}_{A_F}\mathcal{M}$ の射.

$\therefore$ $G: \mathcal{C} \longrightarrow {}_{A_F}\mathcal{M}$ を $G(X) = F(X)$, $G(f) = F(f)$ とすると,

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "G"] \arrow[dr, "F"'] & \arrow[d, phantom, "\circlearrowright"] & {}_{A_F}\mathcal{M} \arrow[dl, "U"] \\
& \mathcal{D} &
\end{tikzcd}$$

$\blacksquare$

## Rem

特に $\mathcal{D} = Vect_k$ のとき, $\theta_k : \underset{\substack{\cong \\ A_F}}{Hom_k(k, A_F)} \cong Nat(k \otimes F, F) = Nat(F, F)$

より $A_F$ と $Nat(F,F)$ は 1対1 に対応する。 $\sigma, \tau \in Nat(k \otimes F, F)$ に対し,

$$\begin{tikzcd}
Hom_k(A_F, A_F) \arrow[r, "(\mu_{A_F} \circ (\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)))^*"] \arrow[d, "\theta_{A_F}"'] & Hom_k(k, A_F) \arrow[d, "\theta_k"] \\
Nat(A_F \otimes F, F) \arrow[r, "(\mu_{A_F} \circ (\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)) \otimes id_F)^*"'] & Nat_k(k \otimes F, F)
\end{tikzcd}$$

$$\begin{tikzcd}
id_{A_F} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & \mu_{A_F} \circ (\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)) \arrow[d, mapsto] \\
\varphi \arrow[r, mapsto] & \varphi \circ (\mu_{A_F} \circ (\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)) \otimes id_F)
\end{tikzcd}$$

$$\begin{tikzcd}
Hom_k(A_F, A_F) \arrow[r, "\theta_k^{-1}(\tau)^*"] \arrow[d, "\theta_{A_F}"'] & Hom_k(k, A_F) \arrow[d, "\theta_k"] \\
Nat(A_F \otimes F, F) \arrow[r, "(\theta_k^{-1}(\tau) \otimes id_F)^*"'] & Nat(k \otimes F, F)
\end{tikzcd}$$

$$\begin{tikzcd}
id_{A_F} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & \theta_k^{-1}(\tau) \arrow[d, mapsto] \\
\varphi \arrow[r, mapsto] & (\theta_k^{-1}(\tau) \otimes id_F) \circ \varphi
\end{tikzcd}$$

$\tau = \theta_k(\theta_k^{-1}(\tau))$

$$\theta_k(\mu_{A_F} \circ (\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau))) = \varphi \circ (\mu_{A_F} \circ (\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)) \otimes id_F) = \varphi \circ (\mu_{A_F} \otimes id_F) \circ ((\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)) \otimes id_F)$$

$$\begin{tikzcd}
(k \otimes k) \otimes F \arrow[r, "a_{k,k,F}"] \arrow[d, "(\theta_k^{-1}(\sigma) \otimes \theta_k^{-1}(\tau)) \otimes id_F"'] \arrow[dr, phantom, "\circlearrowright"] & k \otimes (k \otimes F) \arrow[d, "\theta_k^{-1}(\sigma) \otimes (\theta_k^{-1}(\tau) \otimes id_F)"] \arrow[dr, phantom, "\circlearrowright"] \arrow[r, bend left, "id_k \otimes \tau"] & k \otimes F \arrow[dl, "\theta_k^{-1}(\sigma) \otimes id_F"] \arrow[ddl, bend left, "\sigma"] \\
(A_F \otimes A_F) \otimes F \arrow[r, "a_{A_F,A_F,F}"] \arrow[d, "\mu_{A_F} \otimes id_F"'] \arrow[dr, phantom, "\circlearrowright"] & A_F \otimes (A_F \otimes F) \arrow[d, "id_{A_F} \otimes \varphi"] & \\
A_F \otimes F \arrow[dr, "\varphi"'] & A_F \otimes F \arrow[d, "\varphi"] & \\
& F &
\end{tikzcd}$$

$= \sigma \circ (id_k \otimes \tau) \circ a_{k,k,F}$

左の可換図式から示せる。

$\therefore$ $A_F$ は $Nat(F,F)$ に合成を積と定めた alg と等しいことが分かる。

また単位元は $\theta_k(\eta_{A_F}) = \theta_k(\theta_k^{-1}(l_F)) = id_F$ である。

Vect$_k$ では $id_F$

## Thm

$\mathcal{C}$ : category , $\mathcal{D}$ : locally small monoidal category , $F:\mathcal{C}\longrightarrow\mathcal{D}$ : functor

$Nat(F, F\otimes -):\mathcal{D}\longrightarrow Set$ に対して、$C_F\in\mathcal{D}$ と 自然同型 $\theta: Hom_{\mathcal{D}}(C_F,-)\longrightarrow Nat(F,F\otimes -)$

が存在するとき、以下が成立する。ただし、$\varphi=\theta_{C_F}(id_{C_F})\in Nat(F,F\otimes C_F)$ とする。

(i) ${}^\forall M\in\mathcal{D}$ , ${}^\forall\alpha\in Nat(F,F\otimes M)$ に対して、$\theta_M^{-1}(\alpha)$ は 以下の図式を可換にする unique な射である。

$$\begin{tikzcd}
 & F\otimes M \\
F \arrow[ur, "\alpha"] \arrow[r, "\varphi"'] & F\otimes C_F \arrow[u, "id_F\otimes\theta_M^{-1}(\alpha)"']
\end{tikzcd}$$

(ii) $C_F$ は $\mathcal{D}$ における comonoid str を持つ。

(iii) $U: \mathcal{M}^{C_F}\longrightarrow\mathcal{D}$ を forgetful functor とすると、functor $G:\mathcal{C}\longrightarrow\mathcal{M}^{C_F}$ が存在し、以下の図式は可換となる。

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "G"] \arrow[dr, "F"'] & \circlearrowleft & \mathcal{M}^{C_F} \arrow[dl, "U"] \\
 & \mathcal{D} &
\end{tikzcd}$$

(proof)

P.1 の Thm と dual な議論により示せる。(ii) は $\Delta_{C_F}:=\theta_{C_F\otimes C_F}^{-1}(a_{F,C_F,C_F}\circ(\varphi\otimes id_{C_F})\circ\varphi)$ とすればよい。☒

## Def (monoidal functor)

$(\mathcal{C},\otimes,a,I,l,r)$ , $(\mathcal{C}',\otimes',a',I',l',r')$ : monoidal cat , $F:\mathcal{C}\longrightarrow\mathcal{C}'$ : functor

$J: F(-)\otimes' F(-)\longrightarrow F(-\otimes -)$ : natural iso , $\phi: I'\longrightarrow F(I)$ : isomorphism

このとき $(F,\phi,J)$ を quasi-monoidal functor という。更に以下の (i), (ii) を満たすとき、monoidal functor という。

(i) ${}^\forall X,Y,Z\in Ob(\mathcal{C})$ に対して以下の図式は可換

$$\begin{tikzcd}
(F(X)\otimes' F(Y))\otimes' F(Z) \arrow[r, "J_{X,Y}\otimes id_{F(Z)}"] \arrow[d, "a'_{F(X),F(Y),F(Z)}"] & F(X\otimes Y)\otimes' F(Z) \arrow[r, "J_{X\otimes Y,Z}"] \arrow[d, phantom, "\circlearrowleft"] & F((X\otimes Y)\otimes Z) \arrow[d, "F(a_{X,Y,Z})"] \\
F(X)\otimes'(F(Y)\otimes' F(Z)) \arrow[r, "id_{F(X)}\otimes J_{Y,Z}"'] & F(X)\otimes' F(Y\otimes Z) \arrow[r, "J_{X,Y\otimes Z}"'] & F(X\otimes(Y\otimes Z))
\end{tikzcd}$$

(ii) ${}^\forall X\in Ob(\mathcal{C})$ に対して 以下の図式が可換

$$\begin{tikzcd}
I'\otimes' F(X) \arrow[r, "l'_{F(X)}"] \arrow[d, "\phi\otimes id_{F(X)}"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \\
F(I)\otimes' F(X) \arrow[r, "J_{I,X}"] & F(I\otimes X) \arrow[u, "F(l_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X)\otimes I' \arrow[r, "r_{F(X)}"] \arrow[d, "id_{F(X)}\otimes\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & F(X) \\
F(X)\otimes F(I) \arrow[r, "J_{X,I}"] & F(X\otimes I) \arrow[u, "F(r_X)"']
\end{tikzcd}$$

特に、$\phi=id$, $J=id$ とできるとき、F は strict monoidal functor であるという。

<u>Def</u> (bimonoid, Hopf monoid)

$(\mathcal{C}, \otimes, a, I, l, r, c)$ : braided monoidal cat , $S_H : H \longrightarrow H$ in $\mathcal{C}$

$(B, \mu_B, \eta_B, \Delta_B, \eta_B)$ : bimonoid in $\mathcal{C}$

$\overset{\text{def}}{\Longleftrightarrow}$ $(B, \mu_B, \eta_B)$ : monoid in $\mathcal{C}$ かつ $(B, \Delta_B, \varepsilon_B)$ : comonoid in $\mathrm{Mon}(\mathcal{C})$ ← $\mathcal{C}$ 上の monoid のなす圏

$\Longleftrightarrow$ $(B, \mu_B, \eta_B)$ : monoid in $\mathcal{C}$ かつ $(B, \Delta_B, \varepsilon_B)$ : comonoid in $\mathcal{C}$ かつ $\mu_B, \eta_B$ は comonoid の射

$\Longleftrightarrow$ $(B, \mu_B, \eta_B)$ : monoid in $\mathcal{C}$ かつ $(B, \Delta_B, \varepsilon_B)$ : comonoid in $\mathcal{C}$ かつ $\Delta_B, \varepsilon_B$ は monoid の射

$(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H, S_H)$ : Hopf monoid in $\mathcal{C}$

$\overset{\text{def}}{\Longleftrightarrow}$ $(H, \mu_H, \eta_H, \Delta_H, \varepsilon_H)$ : bimonoid で

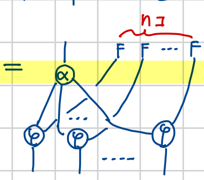
$$\begin{tikzcd}
H \arrow[r, "\varepsilon_H"] \arrow[d, "\Delta_H"'] & I \arrow[r, "\eta_H"] \arrow[d, phantom, "\circlearrowright"] & H \\
H \otimes H \arrow[rr, shift left, "id_H \otimes S_H"] \arrow[rr, shift right, "S_H \otimes id_H"'] & & H \otimes H \arrow[u, "\mu_H"']
\end{tikzcd}$$

<u>Thm</u>

$\mathcal{C}$ : monoidal category , $\mathcal{D}$ : locally small braided monoidal category ,

$F : \mathcal{C} \longrightarrow \mathcal{D}$ : monoidal functor

$\mathrm{Nat}(- \otimes F, F) : \mathcal{D}^{op} \longrightarrow \mathrm{Set}$ に対して, $A_F \in \mathcal{D}$ と 自然同型 $\theta^n : \mathrm{Hom}_{\mathcal{D}}(-, A_F^{\otimes n}) \longrightarrow \mathrm{Nat}(- \otimes F^{\otimes n}, F^{\otimes n})$ $(n \geqq 0)$

が $\varphi = \theta_{A_F}(id_{A_F})$ を用いて $\theta^n(\alpha) =$  の形で与えられているとする。



(以下この条件を representability condition for modules (RAFM) と呼ぶことにする)

が存在するとき, $A_F$ は $\mathcal{D}$ における bimonoid str を持つ。

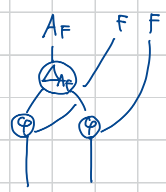
(proof) $\mathrm{Hom}_{\mathcal{D}}(A_F, I) \cong \mathrm{Nat}(A_F \otimes F^0, F^0)$

$A_F$ が $\mathcal{D}$ における alg str を持つことは既に示した。

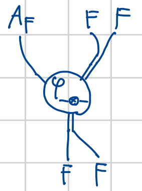
$A_F \otimes F^{\otimes 2} \xrightarrow{id_{A_F} \otimes J_{-,-}} A_F \otimes F(- \otimes -) \xrightarrow{\varphi_{-\otimes -}} F(- \otimes -) \xrightarrow{J^{-1}_{-,-}} F^{\otimes 2}$ $\quad \zeta := J^{-1}_{-,-} \circ \varphi_{-\otimes -} \circ (id_{A_F} \otimes J_{-,-}) \in \mathrm{Nat}(A_F \otimes F^{\otimes 2}, F^{\otimes 2})$

$\Delta_{A_F} := \theta^2_{A_F}(\zeta)$ とする。また, $A_F \xrightarrow{r^{-1}_{A_F}} A_F \otimes I \xrightarrow{id_{A_F} \otimes \phi} A_F \otimes F(I) \xrightarrow{\varphi_I} F(I) \xrightarrow{\phi^{-1}} I$ を $\varepsilon_{A_F}$ とする。

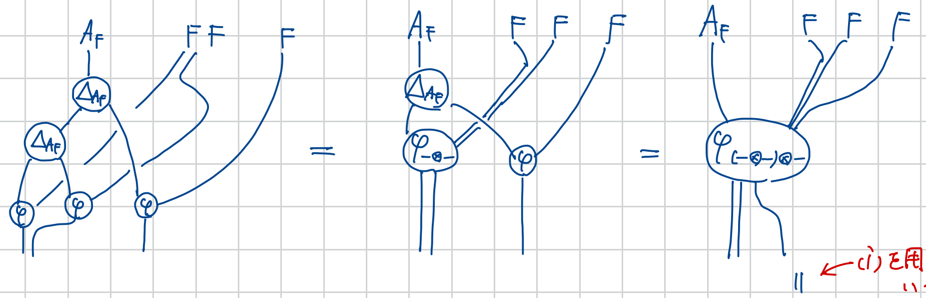
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(A_F^{\otimes 2}, A_F^{\otimes 2}) \arrow[rrr, "\Delta_{A_F}^*"] \arrow[ddd, "\theta^2_{A_F^{\otimes 2}}"'] & & & \mathrm{Hom}_{\mathcal{D}}(A_F, A_F^{\otimes 2}) \arrow[ddd, "\theta^2_{A_F}"] \\
& id_{A_F^{\otimes 2}} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \Delta_{A_F} \arrow[d, mapsto] \arrow[ur, phantom, "\rotatebox{90}{$\in$}"] & \\
& \theta^2_{A_F^{\otimes 2}}(id_{A_F^{\otimes 2}}) \arrow[r, mapsto] \arrow[u, phantom, "\circlearrowright", shift left=6] \arrow[dl, phantom, "\rotatebox{-90}{$\in$}"] & \zeta \arrow[dr, phantom, "\in"] & \\
\mathrm{Nat}(A_F^{\otimes 2} \otimes F^{\otimes 2}, F^{\otimes 2}) \arrow[rrr, "(\Delta_{A_F} \otimes id_{F^{\otimes 2}})^*"'] & & & \mathrm{Nat}(A_F \otimes F^{\otimes 2}, F^{\otimes 2})
\end{tikzcd}$$

$\therefore \ \zeta = \theta^2_{A_F^{\otimes 2}}(id_{A_F^{\otimes 2}}) \circ (\Delta_{A_F} \otimes id_{F^{\otimes 2}}) =$ 

また、$\zeta = J^{-1}_{-,-} \circ \varphi_{-\otimes-} \circ (id_{A_F} \otimes J_{-,-}) =$

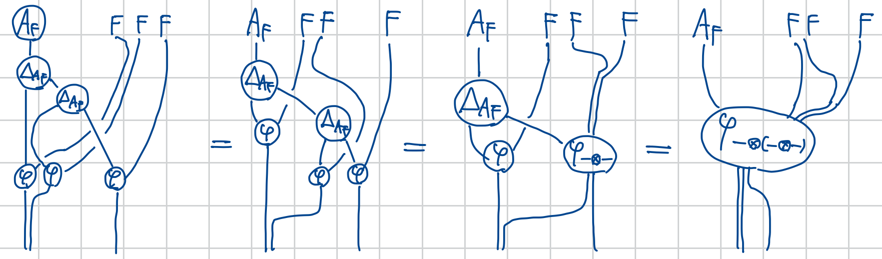


$\theta^n_{A_F}((\Delta_{A_F} \otimes id_{A_F}) \circ \Delta_{A_F}) =$



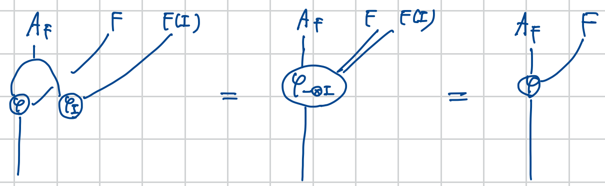
(i)を用いている。

$\theta^n_{A_F}(a^{-1}_{A_F,A_F,A_F} \circ (id_{A_F} \otimes \Delta_{A_F}) \circ \Delta_{A_F}) =$



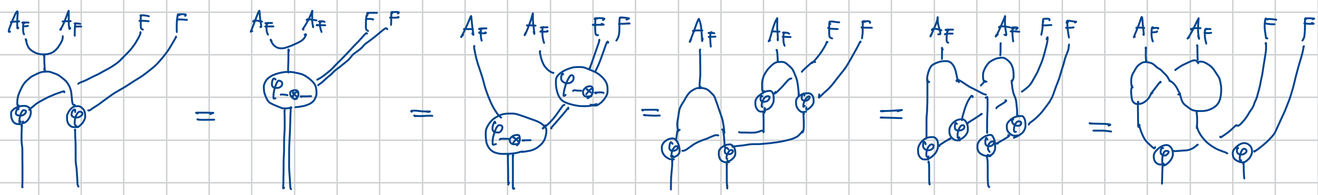
$\therefore\ a_{A_F,A_F,A_F} \circ (\Delta_{A_F} \otimes id_{A_F}) \circ \Delta_{A_F} = (id_{A_F} \otimes \Delta_{A_F}) \circ \Delta_{A_F}$

$\theta_{A_F}((id_{A_F} \otimes \varepsilon_{A_F}) \circ \Delta_{A_F}) =$

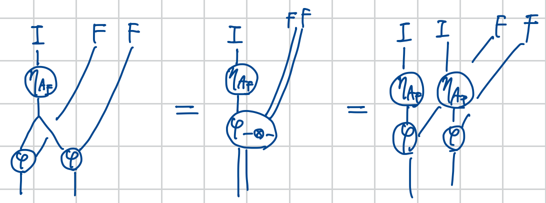


$= \theta_{A_F}(id_{A_F})$

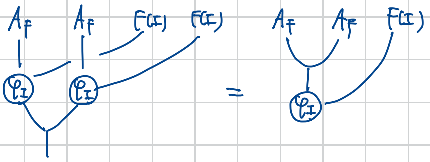
$\therefore\ (id_{A_F} \otimes \varepsilon_{A_F}) \circ \Delta_{A_F} = id_{A_F}$ 全く同様に $(\varepsilon_{A_F} \otimes id_{A_F}) \circ \Delta_{A_F} = id_{A_F}$ (ii)を用いている。



$\therefore\ \theta^2_{A_F^{\otimes 2}}(\Delta_{A_F} \circ \mu_{A_F}) = \theta^2_{A_F^{\otimes 2}}(\mu_{A_F^{\otimes 2}} \circ (\Delta_{A_F} \otimes \Delta_{A_F}))$ $\therefore\ \mu_{A_F^{\otimes 2}} \circ (\Delta_{A_F} \otimes \Delta_{A_F}) = \Delta_{A_F} \circ \mu_{A_F}$

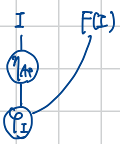


$\therefore\ \Delta_{A_F} \circ \eta_{A_F} = \eta_{A_F^{\otimes 2}}$



$\therefore\ \varepsilon \circ \mu_{A_F} = \underbrace{\mu_I}_{r_I = l_I} \circ (\varepsilon \otimes \varepsilon)$

$\varepsilon_{A_F} \circ \eta_{A_F} =$



$= \theta_I(\eta_{A_F}) = id_F = \eta_I$

これらは、(i)、(ii)を用いていないので、この部分はquasi-monoidalでも成立する。 ▨

## Prop

$V$ : $k$-linear sp. $H$ : f.d. $k$-linear sp. $\mathrm{Hom}_k(V, H\otimes H^*) \cong \mathrm{Hom}_k(V\otimes H, H)$

(proof)

$\mathrm{Hom}_k(V\otimes H, H) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(H,H)) \cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(H, \mathrm{Hom}_k(H^*, k)))$

$\cong \mathrm{Hom}_k(V, \mathrm{Hom}_k(H\otimes H^*, k)) \cong \mathrm{Hom}_k(V, (H\otimes H^*)^*) \cong \mathrm{Hom}_k(V, H\otimes H^*)$

## Rem

上記の対応をexplicitに書くと次のようになる。$\{e_i\}$ : basis of $H$, $\{e^i\}$ : dual basis of $H$

$\Phi : \mathrm{Hom}_k(V, H\otimes H^*) \ni f \longmapsto \Phi(f) \in \mathrm{Hom}_k(V\otimes H, H)$ $\quad \Phi(f)(v\otimes h) = f_2(v)(h) f_1(v)$

$\Psi : \mathrm{Hom}_k(V\otimes H, H) \ni g \longmapsto \Psi(g) \in \mathrm{Hom}_k(V, H\otimes H^*)$ $\quad \Psi(g)(v) = g(v\otimes e_i)\otimes e^i$

$\Psi(\Phi(f))(v) = \Phi(f)(v\otimes e_i)\otimes e^i = f_2(v)(e_i) f_1(v)\otimes e^i = f_1(v)\otimes f_2(v) = f(v)$

$\Phi(\Psi(g))(v\otimes h) = (\Psi(g)_2(v))(h)\, \Psi(g)_1(v) = e^i(h)\, g(v\otimes e_i) = g(v\otimes h)$

## Prop

$V$ : $k$-linear sp. $H$ : f.d. $k$-linear sp. $\mathrm{Hom}_k(H^*\otimes H, V) \cong \mathrm{Hom}_k(H, H\otimes V)$

(proof)

$\{e_i\}$ : basis of $H$, $\{e^i\}$ : dual basis of $H$ とする。

$\Phi' : \mathrm{Hom}_k(H^*\otimes H, V) \ni f \longmapsto \Phi'(f) \in \mathrm{Hom}_k(H, H\otimes V)$ $\quad \Phi'(f)(h) = \sum_i e_i \otimes f(e^i\otimes h)$

$\Psi' : \mathrm{Hom}_k(H, H\otimes V) \ni g \longmapsto \Psi'(g) \in \mathrm{Hom}_k(H^*\otimes H, V)$ $\quad \Psi'(g)(\xi\otimes a) = (\xi\otimes id)(g(a))$

$\Phi'(\Psi'(g))(h) = \sum_i e_i\otimes \Psi'(g)(e^i\otimes h) = \sum_i e_i\otimes (e^i\otimes id)(g(h)) = g(h)$

$\Psi'(\Phi'(f))(\xi\otimes a) = (\xi\otimes id)(\Phi'(f)(a)) = (\xi\otimes id)(\sum_i e_i\otimes f(e^i\otimes a)) = f(\xi\otimes a)$ ▨

**Def** (left-right Hopf module)

$H$ : f.d. Hopf alg, $(M, \lambda_M) \in {}_H\mathcal{M}$, $(M, \rho_M) \in \mathcal{M}^H$

$(M, \lambda_M, \rho_M)$ : left-right Hopf module

$\overset{\text{def}}{\Longleftrightarrow}$ $(h \triangleright m)_0 \otimes (h \triangleright m)_1 = h_1 \triangleright m_0 \otimes h_2 m_1 \quad (\forall h \in H, \forall m \in M)$

$(M, \lambda_M, \rho_M), (N, \lambda_N, \rho_N)$ : left-right Hopf module, $f : M \longrightarrow N$ : $k$-linear map

$f$ : morphism of left-right Hopf modules

$\overset{\text{def}}{\Longleftrightarrow}$ $f$ : morphism of left $H$-modules and right $H$-comodules

また left-right Hopf modules の圏を ${}_H\mathcal{M}^H$ と書く。

**Lemma**

$H$ : f.d. Hopf alg, $-\otimes H : {}_H\mathcal{M} \ni (M, \lambda_M) \longmapsto (M\otimes H, \lambda_{M\otimes H}, \rho_{M\otimes H}) \in {}_H\mathcal{M}^H$ を以下で定める。

$\lambda_{M\otimes H}(h\otimes m\otimes a) = h \triangleright (m\otimes a) = h_1 \triangleright m \otimes h_2 a$, $\rho_{M\otimes H}(m\otimes a) = m\otimes a_1 \otimes a_2$

このとき $-\otimes H$ は functor

(proof)

$$\begin{tikzcd}
H\otimes H\otimes M\otimes H \arrow[r, "id_H\otimes\lambda_{M\otimes H}"] \arrow[ddd, "\mu_H\otimes id_{M\otimes H}"'] & H\otimes M\otimes H \arrow[ddd, "\lambda_{M\otimes H}"] \\
h\otimes h'\otimes m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & h\otimes h'_1\triangleright m\otimes h'_2 a \arrow[d, mapsto] \\
hh'\otimes m\otimes a \arrow[r, mapsto] & h_1h'_1\triangleright m\otimes h_2h'_2 a \\
H\otimes M\otimes H \arrow[r, "\lambda_{M\otimes H}"'] & M\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
k\otimes M\otimes H \arrow[r, "\eta_H\otimes id_{M\otimes H}"] \arrow[ddr, bend right] & H\otimes M\otimes H \arrow[dd, "\lambda_{M\otimes H}"] \\
1_k\otimes m\otimes a \arrow[r, mapsto] \arrow[dr, mapsto, bend right] & 1_H\otimes m\otimes a \arrow[d, mapsto] \\
 & m\otimes a \\
 & M\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes H \arrow[r, "\rho_{M\otimes H}"] \arrow[ddd, "\rho_{M\otimes H}"'] & M\otimes H\otimes H \arrow[ddd, "id_{M\otimes H}\otimes\Delta_H"] \\
m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & m\otimes a_1\otimes a_2 \arrow[d, mapsto] \\
m\otimes a_1\otimes a_2 \arrow[r, mapsto] & m\otimes a_1\otimes a_2\otimes a_3 \\
M\otimes H\otimes H \arrow[r, "\rho_{M\otimes H}\otimes id_H"'] & M\otimes H\otimes H\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes H \arrow[ddd, "\rho_{M\otimes H}"'] \arrow[dddr, bend left] & \\
m\otimes a \arrow[d, mapsto] \arrow[dr, mapsto] & \\
m\otimes a_1\otimes a_2 \arrow[r, mapsto] & m\otimes \varepsilon(a_2)a_1\otimes 1_k = m\otimes a\otimes 1_k \\
M\otimes H\otimes H \arrow[r, "id_{M\otimes H}\otimes\varepsilon_H"'] & M\otimes H\otimes k
\end{tikzcd}$$

$\therefore M\otimes H \in {}_H\mathcal{M}^H$

また $f : M \longrightarrow N$ in ${}_H\mathcal{M}$ に対して $f\otimes id_H : M\otimes H \longrightarrow N\otimes H$ は

$$\begin{tikzcd}
H\otimes M\otimes H \arrow[r, "id_H\otimes f\otimes id_H"] \arrow[ddd, "\lambda_{M\otimes H}"'] & H\otimes N\otimes H \arrow[ddd, "\lambda_{N\otimes H}"] \\
h\otimes m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & h\otimes f(m)\otimes a \arrow[d, mapsto] \\
h_1\triangleright m\otimes h_2 a \arrow[r, mapsto] & f(h_1\triangleright m)\otimes h_2 a \\
M\otimes H \arrow[r, "f\otimes id_H"'] & N\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes H \arrow[r, "f\otimes id_H"] \arrow[ddd, "\rho_{M\otimes H}"'] & N\otimes H \arrow[ddd, "\rho_{N\otimes H}"] \\
m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & f(m)\otimes a \arrow[d, mapsto] \\
m\otimes a_1\otimes a_2 \arrow[r, mapsto] & f(m)\otimes a_1\otimes a_2 \\
M\otimes H\otimes H \arrow[r, "f\otimes id_H\otimes id_H"'] & N\otimes H\otimes H
\end{tikzcd}$$

$\therefore f\otimes id_H$ は ${}_H\mathcal{M}^H$ の morphism

また $(f\circ g)\otimes id_H = (f\otimes id_H)\circ(g\otimes id_H)$

$id_V \otimes id_H = id_{V\otimes H}$

$\therefore -\otimes H$ は functor $\blacksquare$

## Lemma

$H$ : f.d. Hopf alg ， $(M, \lambda_M, \rho_M) \in {}_H\mathcal{M}^H$

このとき $\rho_M$ : morphism of ${}_H\mathcal{M}^H$ （$M \otimes H$ の ${}_H\mathcal{M}^H$ str は先の Lemma で定義したもの）

(proof)

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"'] & M \otimes H \arrow[d, "\rho_{M\otimes H}"] \\
M \otimes H \arrow[r, "\rho_M \otimes id_H"'] & M \otimes H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] & m_0 \otimes m_1 \arrow[d, mapsto] \\
m_0 \otimes m_1 \arrow[r, mapsto] & m_0 \otimes m_1 \otimes m_2
\end{tikzcd}$$

$$\begin{tikzcd}
H \otimes M \arrow[r, "id_H \otimes \rho_M"] \arrow[d, "\lambda_M"'] & H \otimes M \otimes H \arrow[d, "\lambda_{M\otimes H}"] \\
M \arrow[r, "\rho_M"'] & M \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
h \otimes m \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & h \otimes m_0 \otimes m_1 \arrow[d, mapsto] \\
h \triangleright m \arrow[r, mapsto] & h_1 \triangleright m_0 \otimes h_2 m_1 = (h \triangleright m)_0 \otimes (h \triangleright m)_1
\end{tikzcd}$$

$\boxed{\phantom{x}}$

## Prop

$H$ : f.d. Hopf alg， $F : {}_H\mathcal{M}^H \longrightarrow \mathrm{Vect}_k$ : forgetful functor ， $V$ : $k$-vector space

$\mathrm{Nat}(V \otimes F, F) \cong \mathrm{Hom}_k(V \otimes H, H) \cong \mathrm{Hom}_k(V, H \otimes H^*)$

(proof)

$H \in {}_H\mathcal{M}$ より、 $H \otimes H$ は $h \triangleright (a \otimes b) = h_1 a \otimes h_2 b$， $\Delta_{H\otimes H}(a \otimes b) = a \otimes b_1 \otimes b_2$

とすることで ${}_H\mathcal{M}^H$ の object となる。

$\mathrm{Hom}_k(V \otimes H, H) \ni g \longmapsto \tilde{g} \in \mathrm{Nat}(V \otimes F, F)$ を $M \in {}_H\mathcal{M}^H$ に対して

$\tilde{g}_M : V \otimes M \ni v \otimes m \longmapsto g(v \otimes m_1) \triangleright m_0 \in M$ として定める。

$f : M \longrightarrow N$ in ${}_H\mathcal{M}^H$ に対して

$$\left(\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\rho_M"'] & N \arrow[d, "\rho_N"] \\
M \otimes H \arrow[r, "f \otimes id_H"'] & N \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\textcircled{1}\ \circlearrowright"] & f(m) \arrow[d, mapsto] \\
m_0 \otimes m_1 \arrow[r, mapsto] & f(m_0) \otimes m_1 = f(m)_0 \otimes f(m)_1
\end{tikzcd}\right)$$

$$\begin{tikzcd}
V \otimes M \arrow[r] \arrow[d, "\tilde{g}_M"'] & V \otimes N \arrow[d, "\tilde{g}_N"] \\
M \arrow[r] & N
\end{tikzcd}
\qquad
\begin{tikzcd}
v \otimes m \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & v \otimes f(m) \arrow[d, mapsto] \\
g(v \otimes m_1) \triangleright m_0 \arrow[r, mapsto] & g(v \otimes m_1) \triangleright f(m_0) \overset{\textcircled{1}}{=} g(v \otimes f(m)_1) \triangleright f(m)_0
\end{tikzcd}$$

$\therefore$ $\tilde{g}$ は naturality をみたす。

また、 $\mathrm{Nat}(V \otimes F, F) \ni \tau \longmapsto \check{\tau} = \left(v \otimes h \longmapsto (id_H \otimes \varepsilon_H)(\tau_{H\otimes H}(v \otimes 1_H \otimes h))\right) \in \mathrm{Hom}_k(V \otimes H, H)$

$g \in \mathrm{Hom}_k(V \otimes H, H)$ に対して

$\check{\tilde{g}}(v \otimes h) = (id_H \otimes \varepsilon_H)(\tilde{g}_{H\otimes H}(v \otimes 1_H \otimes h)) = (id_H \otimes \varepsilon_H)(g(v \otimes h_2) \triangleright (1_H \otimes h_1)) = (id_H \otimes \varepsilon_H)(g(v \otimes h_2)_1 \otimes g(v \otimes h_2)_2 h_1)$

$= \varepsilon(h_1) g(v \otimes h_2) = g(v \otimes h)$ $\qquad \therefore \check{\tilde{g}} = g$

$M \in {}_H\mathcal{M}^H$, $m \in M$ に対して，$- \triangleright m : H \ni h \longmapsto h \triangleright m \in M$ とすると $(-\triangleright m)(hh') = hh' \triangleright m = h \triangleright (-\triangleright m)(h')$

$\therefore$ $-\triangleright m$ は ${}_H\mathcal{M}$ の morphism $\therefore$ $(-\triangleright m) \otimes id_H : H \otimes H \longrightarrow M \otimes H$ は ${}_H\mathcal{M}^H$ の morphism

$\therefore$ $\tau \in Nat(V \otimes F, F)$ に対して

$$\begin{tikzcd}
V \otimes H \otimes H \arrow[r, "id_V \otimes (-\triangleright m) \otimes id_H"] \arrow[d, "\tau_{H\otimes H}"'] & V \otimes M \otimes H \arrow[d, "\tau_{M\otimes H}"] \\
H \otimes H \arrow[r, "(-\triangleright m) \otimes id_H"'] & M \otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
v \otimes h \otimes h' \arrow[r, mapsto] \arrow[d, mapsto] & v \otimes h \triangleright m \otimes h' \arrow[d, mapsto] \\
\tau_{H\otimes H}(v \otimes h \otimes h') \arrow[r, mapsto] & (-\triangleright m)(\tau_{H\otimes H}(v \otimes h \otimes h'))
\end{tikzcd}$$

特に $h = 1_H$ のとき

$$\tau_{M\otimes H}(v \otimes m \otimes h') = ((-\triangleright m) \otimes id_H)(\tau_{H\otimes H}(v \otimes 1_H \otimes h'))$$

また，$\rho_M : M \longrightarrow M \otimes H$ が ${}_H\mathcal{M}^H$ の morphism より

$$\begin{tikzcd}
V \otimes M \arrow[r, "id_V \otimes \rho_M"] \arrow[d, "\tau_M"'] & V \otimes M \otimes H \arrow[d, "\tau_{M\otimes H}"] \\
M \arrow[r, "\rho_M"] \arrow[dr, "id_M"'] & M \otimes H \arrow[d, "id_M \otimes \varepsilon_H"] \\
 & M
\end{tikzcd}$$

$$\begin{tikzcd}
v \otimes m \arrow[r, mapsto] \arrow[d, mapsto] & v \otimes m_0 \otimes m_1 \arrow[d, mapsto] \\
\tau_M(v \otimes m) \arrow[r, mapsto] & \tau_{M\otimes H}(v \otimes m_0 \otimes m_1) = \tau_M(v \otimes m)_0 \otimes \tau_M(v \otimes m)_1
\end{tikzcd}$$

$$\tilde{\check{\tau}}_M(v \otimes m) = \check{\tau}(v \otimes m_1) \triangleright m_0 = (id_H \otimes \varepsilon_H)(\tau_{H\otimes H}(v \otimes 1_H \otimes m_1)) \triangleright m_0 = ((-\triangleright m_0) \circ (id_H \otimes \varepsilon_H) \circ \tau_{H\otimes H})(v \otimes 1_H \otimes m_1)$$

$$= ((id_H \otimes \varepsilon_H) \circ ((-\triangleright m_0) \otimes id_H) \circ \tau_{H\otimes H})(v \otimes 1_H \otimes m_1) = (id_H \otimes \varepsilon_H)(\tau_{M\otimes H}(v \otimes m_0 \otimes m_1)) = ((id_H \otimes \varepsilon_H) \circ \tau_{M\otimes H} \circ (id_V \otimes \Delta_M))(v \otimes m)$$

$= \tau_M(v \otimes m)$ $\quad \therefore \tilde{\check{\tau}} = \tau \quad \therefore Nat(V \otimes F, F) \cong Hom_k(V \otimes H, H)$

ここで，$\otimes$ と Hom は Adjoint より，$Hom_k(V \otimes H, H) \cong Hom_k(V, Hom_k(H, H))$

$\therefore$ $Nat(- \otimes F, F) \cong Hom_k(-, Hom_k(H, H))$

ここで $Hom_k(H, H) \cong Hom_k(H, Hom_k(H^*, k)) \cong Hom_k(H \otimes H^*, k) \cong (H \otimes H^*)^* \cong H \otimes H^*$

$\therefore$ $Nat(- \otimes F, F) \cong Hom_k(-, H \otimes H^*)$ ▨

## Rem

$\Theta_V : Hom_k(V, H \otimes H^*) \xrightarrow{\Phi} Hom_k(V \otimes H, H) \xrightarrow{\sim} Nat(V \otimes F, F)$

$\Theta_V^{-1} : Nat(V \otimes F, F) \xrightarrow{\vee} Hom_k(V \otimes H, H) \xrightarrow{\Phi} Hom_k(V, H \otimes H^*)$ を明示的に表すと

$f \in Hom_k(V, H \otimes H^*)$, $M \in {}_H\mathcal{M}^H$, $m \in M$, $v \in V$ に対し

$$\Theta_V(f)_M(v \otimes m) = \widetilde{\Phi(f)}_M(v \otimes m) = \Phi(f)(v \otimes m_1) \triangleright m_0 = f_2(v)(m_1) f_1(v) \triangleright m_0$$

$\tau \in Nat(V \otimes F, F)$, $v \in V$ とする。

$$\Theta_V^{-1}(\tau)(v) = \Phi(\check{\tau})(v) = \check{\tau}(v \otimes e_i) \otimes e^i = (id_H \otimes \varepsilon_H)(\tau_{H\otimes H}(v \otimes 1_H \otimes e_i)) \otimes e^i$$

Prop

上記の $H\otimes H^*$ の積構造は以下の形で与えられる。

$$(a\# \xi)\cdot(b\# \nu) = ab_1 \# (\xi \leftharpoonup b_2) * \nu$$

また、単位元は $1_H \# \varepsilon_H$ である。

(proof)

$a\otimes\xi,\ b\otimes\nu \in H\otimes H^*,\ M\in {}_H\mathcal{M}^H,\ m\in M$ とする

$$\varphi_M(a\otimes\xi\otimes m) = \theta_{H\otimes H^*}(id_{H\otimes H^*})_M(a\otimes\xi\otimes m) = \xi(m_1)\, a \triangleright m_0$$

$$(\varphi\circ(id\otimes\varphi))_M(a\otimes\xi\otimes b\otimes\nu\otimes m) = (\varphi_M\circ(id\otimes\varphi_M))(a\otimes\xi\otimes b\otimes\nu\otimes m) = \varphi_M(a\otimes\xi\otimes\nu(m_1) b\triangleright m_0)$$

$$= \nu(\underbrace{m_1}_{m_2})\,\xi(\underbrace{(b\triangleright m_0)_1}_{b_2\triangleright \underbrace{(m_0)_1}_{m_1}})\, a\triangleright \underbrace{(b\triangleright m_0)_0}_{b_1\triangleright\underbrace{(m_0)_0}_{m_0}} = ((\xi \leftharpoonup b_2)*\nu)(m_1)\, ab_1\triangleright m_0$$

$$\theta_{H\otimes H^*\otimes H\otimes H^*}^{-1}(\varphi\circ(id\otimes\varphi))(a\otimes\xi\otimes b\otimes\nu) = (id_H\otimes\varepsilon_H)\cdot((\varphi\circ(id\otimes\varphi))_{H\otimes H}(a\otimes\xi\otimes b\otimes\nu\otimes 1_H\otimes e_i)\otimes e^i$$

$$= (id_H\otimes\varepsilon_H)\left(((\xi\leftharpoonup b_2)*\nu)(e_{i_2})(ab_1)\triangleright(1_H\otimes e_{i_1})\right)\otimes e^i = (id_H\otimes\varepsilon_H)\left(((\xi\leftharpoonup b_2)*\nu)(e_{i_2})\, a_1b_{11}\otimes a_2b_{12}e_{i_1}\right)\otimes e^i$$

$$= ab_1\otimes(\xi\leftharpoonup b_2)*\nu$$

$$\eta_{H\otimes H^*}(1_{\Bbbk}) = \theta_{\Bbbk}^{-1}(l_F)(1_{\Bbbk}) = (id_H\otimes\varepsilon_H)(l_{H\otimes H}(1_{\Bbbk}\otimes 1_H\otimes e_i))\otimes e^i = (id_H\otimes\varepsilon_H)(1_H\otimes e_i)\otimes e^i = 1_H\otimes\varepsilon_H \quad ▨$$

## Prop

$H$ : f.d. Hopf alg, $F : {}_H\mathcal{M}^H \longrightarrow \mathrm{Vect}_{\Bbbk}$ : forgetful functor, $V$ : $\Bbbk$-vector space

$$\mathrm{Nat}(F, F\otimes V) \cong \mathrm{Hom}_{\Bbbk}(H, H\otimes V) \cong \mathrm{Hom}_{\Bbbk}(H^*\otimes H, V)$$

(proof)

$H \in {}_H\mathcal{M}$ より、$H\otimes H$ は $h \triangleright (a\otimes b) = h_1 a \otimes h_2 b$, $\rho_{H\otimes H}(a\otimes b) = a\otimes b_1 \otimes b_2$

とすることで ${}_H\mathcal{M}^H$ の object となる。

$\mathrm{Hom}_{\Bbbk}(H, H\otimes V) \ni g \longmapsto \tilde{g} \in \mathrm{Nat}(F, F\otimes V)$ を $M \in {}_H\mathcal{M}^H$ に対して

$\tilde{g}_M : M \ni m \mapsto (-\triangleright m_0 \otimes id_V)(g(m_1)) \in M\otimes V$ として定める。

$f : M \longrightarrow N$ in ${}_H\mathcal{M}^H$ に対して

$$\left(\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\rho_M"'] & N \arrow[d, "\rho_N"] \\
M\otimes H \arrow[r, "f\otimes id_H"'] & N\otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "① \circlearrowleft"] & f(m) \arrow[d, mapsto] \\
m_0\otimes m_1 \arrow[r, mapsto] & f(m)_0\otimes f(m)_1 = f(m_0)\otimes m_1 \in N\otimes H
\end{tikzcd}\right)$$

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\tilde{g}_M"'] & N \arrow[d, "\tilde{g}_N"] \\
M\otimes V \arrow[r, "f\otimes id_V"'] & N\otimes V
\end{tikzcd}
\qquad
\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & f(m) \arrow[d, mapsto] \\
(-\triangleright m_0\otimes id_V)(g(m_1)) \arrow[r, mapsto] & (-\triangleright f(m)_0\otimes id_V)(g(f(m)_1)) = (-\triangleright f(m_0)\otimes id_V)(g(m_1))
\end{tikzcd}$$

$\therefore \tilde{g}$ は naturality をみたす。

また、$\mathrm{Nat}(F, F\otimes V) \ni \tau \longmapsto \check{\tau} = \left(h \longmapsto (id_H\otimes \varepsilon_H \otimes id_V)(\tau_{H\otimes H}(1_H\otimes h))\right) \in \mathrm{Hom}_{\Bbbk}(H, H\otimes V)$

$g \in \mathrm{Hom}_{\Bbbk}(H, H\otimes V)$ に対して $\rho_{H\otimes H}(1_H\otimes h) = 1_H \otimes h_1 \otimes h_2$

$$\check{\tilde{g}}(h) = (id_H\otimes \varepsilon_H\otimes id_V)\circ(\tilde{g}_{H\otimes H}(1_H\otimes h)) = (id_H\otimes \varepsilon_H\otimes id_V)(-\triangleright(1_H\otimes h_1)\otimes id_V)(\underbrace{g(h_2)}_{g_1(h_2)\otimes g_2(h_2)})$$

$$= (id_H\otimes \varepsilon_H\otimes id_V)(g_1(h_2)_1 \otimes g_1(h_2)_2 h_1 \otimes g_2(h_2)) = g_1(h)\otimes g_2(h) = g(h)$$

$$\therefore \check{\tilde{g}} = g$$

$M\in {}_H\mathcal{M}^H$, $m\in M$ に対して、$-\triangleright m : H\ni h\longmapsto h\triangleright m\in M$ は

$$\begin{tikzcd}
H\otimes H \arrow[r, "id_H\otimes(-\triangleright m)"] \arrow[d, "\mu_H=\lambda_H"'] \arrow[dr, phantom, "\circlearrowleft"] & H\otimes M \arrow[d, "\lambda_M"] \\
H \arrow[r, "(-\triangleright m)"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
h\otimes a \arrow[r, mapsto] \arrow[d, mapsto] & h\otimes a\triangleright m \arrow[d, mapsto] \\
 & h\triangleright(a\triangleright m) \arrow[d, phantom, "\shortparallel"] \\
ha \arrow[r, mapsto] & ha\triangleright m
\end{tikzcd}$$

より $-\triangleright m$ は ${}_H\mathcal{M}$ の morphism

$\therefore\ (-\triangleright m)\otimes id_H : H\otimes H\longrightarrow M\otimes H$ は ${}_H\mathcal{M}^H$ の morphism

$\therefore\ \tau\in Nat(F, F\otimes V)$ に対して、

$$\begin{tikzcd}
H\otimes H \arrow[r, "(-\triangleright m)\otimes id_H"] \arrow[d, "\tau_{H\otimes H}"'] & M\otimes H \arrow[d, "\tau_{M\otimes H}"] \\
H\otimes H\otimes V \arrow[r, "(-\triangleright m)\otimes id_H\otimes id_V"'] & M\otimes H\otimes V
\end{tikzcd}$$

$$\begin{tikzcd}
h\otimes h' \arrow[r, mapsto] \arrow[d, mapsto] & h\triangleright m\otimes h' \arrow[d, mapsto] \\
 & \tau_{M\otimes H}(h\triangleright m\otimes h') \arrow[d, phantom, "\shortparallel"] \\
\tau_{H\otimes H}(h\otimes h') \arrow[r, mapsto] & ((-\triangleright m)\otimes id_H\otimes id_V)(\tau_{H\otimes H}(h\otimes h'))
\end{tikzcd}$$

特に $h=1_H$ のとき、

$$((-\triangleright m)\otimes id_H\otimes id_V)(\tau_{H\otimes H}(1\otimes h'))=\tau_{M\otimes H}(m\otimes h')$$

また、$\rho_M : M\longrightarrow M\otimes H$ が ${}_H\mathcal{M}^H$ の morphism より

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[d, "\tau_M"'] & M\otimes H \arrow[d, "\tau_{M\otimes H}"] \\
M\otimes V \arrow[r, "\rho_M\otimes id_V"'] \arrow[dr, "id_M\otimes id_V"'] \arrow[r, phantom, bend right=60, "\circlearrowleft"] & M\otimes H\otimes V \arrow[d, "id_M\otimes\varepsilon_H\otimes id_V"] \\
 & M\otimes V
\end{tikzcd}$$

$$\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] & m_0\otimes m_1 \arrow[d, mapsto] \\
\tau_M(m) \arrow[r, mapsto] & (\rho_M\otimes id_V)(\tau_M(m))=\tau_{M\otimes H}(m_0\otimes m_1)
\end{tikzcd}$$

$$\tilde{\tau}_M(m)=(-\triangleright m_0\otimes id_V)(\check{\tau}(m_1))=((-\triangleright m_0\otimes id_V)\circ(id_H\otimes\varepsilon_H\otimes id_V))(\tau_{H\otimes H}(1_H\otimes m_1))$$

$$=((id_M\otimes\varepsilon_H\otimes id_V)\circ(-\triangleright m_0\otimes id_H\otimes id_V))(\tau_{H\otimes H}(1_H\otimes m_1))=(id_M\otimes\varepsilon_H\otimes id_V)(\tau_{M\otimes H}(m_0\otimes m_1))=\tau_M(m)$$

$\therefore\ \tilde{\tau}=\tau \qquad \therefore\ Nat(F, F\otimes V)\cong Hom_k(H, H\otimes V)$

また、$\{e_i\}$ : basis of $H$, $\{e^i\}$ : dual basis とすると、

$$\Phi : Hom_k(H^*\otimes H, V)\ni\varphi\longmapsto\Phi(\varphi)=\left(h\longmapsto\sum_i e_i\otimes\varphi(e^i\otimes h)\right)\in Hom_k(H, H\otimes V)$$

$$\Psi : Hom_k(H, H\otimes V)\ni\psi\longmapsto\Psi(\psi)=\left(\xi\otimes a\longmapsto(\xi\otimes id)(\psi(a))\right)\in Hom_k(H^*\otimes H, V)$$

$$((\Phi\circ\Psi)(\psi))(h)=\Phi(\Psi(\psi))(h)=\sum_i e_i\otimes\Psi(\psi)(e^i\otimes h)=\sum_i e_i\otimes(e^i\otimes id)(\psi(h))=\psi(h)$$

$$((\Psi\circ\Phi)(\varphi))(\xi\otimes a)=\Psi(\Phi(\varphi))(\xi\otimes a)=(\xi\otimes id)(\Phi(\varphi)(a))=(\xi\otimes id)\left(\sum_i e_i\otimes\varphi(e^i\otimes a)\right)=\varphi(\xi\otimes a)$$

$\therefore\ Hom_k(H, H\otimes V)\cong Hom_k(H^*\otimes H, V)$ ▨

## Rem

$\theta_V : \mathrm{Hom}_{\mathbb{k}}(H^*\otimes H, V) \xrightarrow{\Phi} \mathrm{Hom}_{\mathbb{k}}(H, H\otimes V) \xrightarrow{\sim} \mathrm{Nat}(F, F\otimes V)$

$\theta_V^{-1} : \mathrm{Nat}(F, F\otimes V) \xrightarrow{\vee} \mathrm{Hom}_{\mathbb{k}}(H, H\otimes V) \xrightarrow{\Phi} \mathrm{Hom}_{\mathbb{k}}(H^*\otimes H, V)$ を明示的に表すと

$f \in \mathrm{Hom}_{\mathbb{k}}(H^*\otimes H, V)$, $M \in {}_H\mathcal{M}^H$, $m \in M$ に対し

$\theta_V(f)_M(m) = \widetilde{\Phi(f)}_M(m) = (-\triangleright m_0 \otimes id_V)(\Phi(f)(m_1)) = (-\triangleright m_0 \otimes id_V)\left(\sum_i e_i \otimes f(e^i\otimes m_1)\right) = \sum_i e_i \triangleright m_0 \otimes f(e^i \otimes m_1)$

$\tau \in \mathrm{Nat}(F, F\otimes V)$, $\xi \otimes a \in H^*\otimes H$ に対し

$\theta_V^{-1}(\tau)(\xi\otimes a) = \Phi(\check{\tau})(\xi\otimes a) = (\xi\otimes id_V)(\check{\tau}(a)) = (\xi \otimes id_V)((id_H\otimes \varepsilon_H \otimes id_H)(\tau_{H\otimes H}(1_H\otimes a)))$

## Prop

上記の $H^*\otimes H$ の余積構造は以下の形で与えられる。

$\Delta(\xi\otimes a) = \xi_1 \otimes e_i a_1 \otimes \xi_2 * e^i \otimes a_2$

(proof)

$M \in {}_H\mathcal{M}^H$, $m \in M$ とする。

$\varphi_M(m) = \theta_{H^*\otimes H}(id_{H^*}\otimes id_H)_M(m) = \sum_i e_i \triangleright m_0 \otimes e^i \otimes m_1$

$((\varphi\otimes id_{H^*\otimes H})\circ\varphi)_M(m) = (\varphi\otimes id_{H^*\otimes H})\left(\sum_i e_i \triangleright m_0 \otimes e^i \otimes m_1\right) = \sum_{i,j} e_j \triangleright \underset{e_{i_1}\triangleright \underset{m_0}{(m_0)_0}}{(e_i\triangleright m_0)_0} \otimes e^j \otimes \underset{e_{i_2}\underset{m_1}{(m_0)_1}}{(e_i\triangleright m_0)_1} \otimes e^i \otimes \underset{m_2}{m_1}$

$= \sum_{i,j} e_j e_{i_1} \triangleright m_0 \otimes e^j \otimes e_{i_2} m_1 \otimes e^i \otimes m_2$

$\Delta_{H^*\otimes H}(\xi\otimes a) = \theta^{-1}_{H^*\otimes H\otimes H^*\otimes H}((\varphi\otimes id_{H^*\otimes H})\circ\varphi)(\xi\otimes a) = (\xi\otimes id_{H^*\otimes H\otimes H^*\otimes H})((id\otimes\varepsilon\otimes id)((\varphi\otimes id)\circ\varphi)_{H\otimes H}(1_H\otimes a)))$

$= (\xi\otimes id_{H^*\otimes H\otimes H^*\otimes H})(id_H\otimes\varepsilon_H\otimes id_{H^*\otimes H\otimes H^*\otimes H})\left(\sum_{i,j} \underset{\underset{e_j e_{i_1}}{e_{j_1}e_{i_{11}}\otimes e_{j_2}e_{i_{12}}a_1}}{e_j e_{i_1}\triangleright(1_H\otimes a_1)} \otimes e^j \otimes e_{i_2}\underset{a_1}{a_2} \otimes e^i \otimes \underset{a_2}{a_3}\right) = \sum_{i,j} \underset{\xi_1(e_j)\,\xi_2(e_{i_1})}{\xi(e_j e_{i_1})}\, \overset{\xi_1}{e^j} \otimes e_{i_2}a_1 \otimes e^i \otimes a_2$

$= \sum_i \xi_1 \otimes (\xi_2 * id)(e_i) a_1 \otimes e^i \otimes a_2 = \sum_i \xi_1 \otimes (e_i \leftharpoonup \xi_2) a_1 \otimes e^i \otimes a_2 = \sum_i \xi_1 \otimes e_i a_1 \otimes \xi_2 * e^i \otimes a_2$ ▨

## Rem

$F : {}_H\mathcal{M}^H \longrightarrow \mathrm{Vect}_{\mathbb{k}}$ : forgetful functor に対して $A_F \cong (C_F)^*$ である。

$a\otimes\xi,\ b\otimes\nu \in (C_F)^* = (H^*\otimes H)^* = H\otimes H^*$

$(\mu_{(C_F)^*}(a\otimes\xi\otimes b\otimes\nu))(\zeta\otimes c) = ((a\otimes\xi\otimes b\otimes\nu)\circ\Delta_{C_F})(\zeta\otimes c) = (a\otimes\xi\otimes b\otimes\nu)\left(\sum_i \zeta_1\otimes e_i c_1 \otimes \zeta_2 * e^i \otimes c_2\right)$

$= \sum_i \zeta_1(a)\,\xi(\underset{b_2}{e_i} c_1)\,\zeta_2(b_1)\,e^i(b_2)\,\nu(c_2) = \zeta(ab_1)((\xi\leftharpoonup b_2)*\nu)(c) = (ab_1\otimes(\xi\leftharpoonup b_2)*\nu)(\zeta\otimes c)$

$\therefore\ \mu_{A_F} = \mu_{(C_F)^*}$ $\quad\therefore\ A_F \cong (C_F)^*$

## Def (left-right YD module)

$H$ : f.d. Hopf alg, $(M, \lambda_M) \in {}_H\mathcal{M}$, $(M, \rho_M) \in \mathcal{M}^H$

$(M, \lambda_M, \rho_M)$ : left-right YD module

$\overset{\text{def}}{\iff}$ $h_1 \triangleright m_0 \otimes h_2 m_1 = (h_2 \triangleright m)_0 \otimes (h_2 \triangleright m)_1 h_1$ $\quad ({}^\forall h \in H, {}^\forall m \in M)$

$(h \triangleright m)_0 \otimes (h \triangleright m)_1 = (h_3 \triangleright m)_0 \otimes (h_3 \triangleright m)_1 h_2 S^{-1}(h_1) = h_2 \triangleright m_0 \otimes h_3 m_1 S^{-1}(h_1)$

$(M, \lambda_M, \rho_M), (N, \lambda_N, \rho_N)$ : left-right YD module, $f : M \longrightarrow N$ : $\Bbbk$-linear map

$f$ : morphism of left-right YD modules

$\overset{\text{def}}{\iff}$ $f$ : morphism of left $H$-modules and right $H$-comodules

また left-right YD modules の圏を ${}_H\mathcal{YD}^H$ と書く。

## Lemma

$H$ : f.d. Hopf alg, $-\otimes H : {}_H\mathcal{M} \ni (M, \lambda_M) \longmapsto (M\otimes H, \lambda_{M\otimes H}, \rho_{M\otimes H}) \in {}_H\mathcal{YD}^H$ を以下で定める。

$\lambda_{M\otimes H}(h\otimes m\otimes a) = h \triangleright (m\otimes a) = h_2 \triangleright m \otimes h_3 a S^{-1}(h_1)$, $\rho_{M\otimes H}(m\otimes a) = m\otimes a_1 \otimes a_2$

このとき $-\otimes H$ は functor

(proof)

$$\begin{tikzcd}
H\otimes H\otimes M\otimes H \arrow[r, "id_H\otimes\lambda_{M\otimes H}"] \arrow[d, "\mu_H\otimes id_{M\otimes H}"'] & H\otimes M\otimes H \arrow[d, "\lambda_{M\otimes H}"] \\
H\otimes M\otimes H \arrow[r, "\lambda_{M\otimes H}"'] & M\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
h\otimes h'\otimes m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & h\otimes h'_2\triangleright m\otimes h'_3 a S^{-1}(h'_1) \arrow[d, mapsto] \\
hh'\otimes m\otimes a \arrow[r, mapsto] & (hh')_2\triangleright m\otimes (hh')_3 a S^{-1}((hh')_1)
\end{tikzcd}$$

$$\begin{tikzcd}
\Bbbk\otimes M\otimes H \arrow[r, "\eta_H\otimes id_{M\otimes H}"] \arrow[dr, bend right] & H\otimes M\otimes H \arrow[d, "\lambda_{M\otimes H}"] \\
 & M\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
1_\Bbbk\otimes m\otimes a \arrow[r, mapsto] \arrow[dr, mapsto, bend right] & 1_H\otimes m\otimes a \arrow[d, mapsto] \arrow[dl, phantom, "\circlearrowright"] \\
 & m\otimes a
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes H \arrow[r, "\rho_{M\otimes H}"] \arrow[d, "\rho_{M\otimes H}"'] & M\otimes H\otimes H \arrow[d, "id_{M\otimes H}\otimes\Delta_H"] \\
M\otimes H\otimes H \arrow[r, "\rho_{M\otimes H}\otimes id_H"'] & M\otimes H\otimes H\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & m\otimes a_1\otimes a_2 \arrow[d, mapsto] \\
m\otimes a_1\otimes a_2 \arrow[r, mapsto] & m\otimes a_1\otimes a_2\otimes a_3
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes H \arrow[d, "\rho_{M\otimes H}"'] \arrow[dr, bend left] & \\
M\otimes H\otimes H \arrow[r, "id_{M\otimes H}\otimes\varepsilon_H"'] & M\otimes H\otimes\Bbbk
\end{tikzcd}$$

$$\begin{tikzcd}
m\otimes a \arrow[d, mapsto] \arrow[dr, mapsto] & \\
m\otimes a_1\otimes a_2 \arrow[r, mapsto] \arrow[ur, phantom, "\circlearrowright"] & m\otimes\varepsilon(a_2)a_1\otimes 1_\Bbbk = m\otimes a\otimes 1_\Bbbk
\end{tikzcd}$$

$(h\triangleright(m\otimes a))_0 \otimes (h\triangleright(m\otimes a))_1$
$= h_3\triangleright m\otimes h_4 a_1 S^{-1}(h_2)\otimes h_5 a_2 S^{-1}(h_1)$
$= h_2\triangleright(m\otimes a)_0 \otimes h_3 (m\otimes a)_1 S^{-1}(h_1)$

$\therefore M\otimes H \in {}_H\mathcal{YD}^H$

また $f : M \longrightarrow N$ in ${}_H\mathcal{M}$ に対して $f\otimes id_H : M\otimes H \longrightarrow N\otimes H$ は

$$\begin{tikzcd}
H\otimes M\otimes H \arrow[r, "id_H\otimes f\otimes id_H"] \arrow[d, "\lambda_{M\otimes H}"'] & H\otimes N\otimes H \arrow[d, "\lambda_{N\otimes H}"] \\
M\otimes H \arrow[r, "f\otimes id_H"'] & N\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
h\otimes m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & h\otimes f(m)\otimes a \arrow[d, mapsto] \\
h_2\triangleright m\otimes h_3 a S^{-1}(h_1) \arrow[r, mapsto] & h_2\triangleright f(m)\otimes h_3 a S^{-1}(h_1)
\end{tikzcd}$$

$$\begin{tikzcd}
M\otimes H \arrow[r, "f\otimes id_H"] \arrow[d, "\rho_{M\otimes H}"'] & N\otimes H \arrow[d, "\rho_{N\otimes H}"] \\
M\otimes H\otimes H \arrow[r, "f\otimes id_H\otimes id_H"'] & N\otimes H\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
m\otimes a \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & f(m)\otimes a \arrow[d, mapsto] \\
m\otimes a_1\otimes a_2 \arrow[r, mapsto] & f(m)\otimes a_1\otimes a_2
\end{tikzcd}$$

$\therefore f\otimes id_H$ は ${}_H\mathcal{YD}^H$ の morphism

また $(f\circ g)\otimes id_H = (f\otimes id_H)\circ(g\otimes id_H)$

$id_V \otimes id_H = id_{V\otimes H}$

$\therefore -\otimes H$ は functor ▨

## Lemma

$H$ : f.d. Hopf alg , $(M,\lambda_M,\rho_M)\in {}_H\mathcal{YD}^H$

このとき $\rho_M$ : morphism of ${}_H\mathcal{YD}^H$ ( $M\otimes H$ の ${}_H\mathcal{YD}^H$ str は先の lemma で定義したもの )

(proof)

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[d, "\rho_M"'] & M\otimes H \arrow[d, "\rho_{M\otimes H}"] \\
M\otimes H \arrow[r, "\rho_M\otimes id_H"'] & M\otimes H\otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
H\otimes M \arrow[r, "id_H\otimes\rho_M"] \arrow[d, "\lambda_M"'] & H\otimes M\otimes H \arrow[d, "\lambda_{M\otimes H}"] \\
M \arrow[r, "\rho_M"'] & M\otimes H
\end{tikzcd}$$

$m\longmapsto m_0\otimes m_1\longmapsto m_0\otimes m_1\otimes m_2$, $\quad m\longmapsto m_0\otimes m_1\longmapsto m_0\otimes m_1\otimes m_2$

$h\otimes m\longmapsto h\otimes m_0\otimes m_1 \longmapsto h_2\triangleright m_0\otimes h_3 m_1 S^{-1}(h_1) = (h\triangleright m)_0\otimes(h\triangleright m)_1$

$h\otimes m\longmapsto h\triangleright m\longmapsto (h\triangleright m)_0\otimes(h\triangleright m)_1$ $\quad\blacksquare$

## Prop

$H$ : f.d. Hopf alg , $F: {}_H\mathcal{YD}^H\longrightarrow Vect_k$ : forgetful functor , $V$ : $k$-vector space

$Nat(V\otimes F, F)\cong Hom_k(V\otimes H, H)\cong Hom_k(V, End_k(H))$

(proof)

$H\in {}_H\mathcal{M}$ より、 $H\otimes H$ は $h\triangleright(a\otimes b) = h_2 a\otimes h_3 b S^{-1}(h_1)$ , $\Delta_{H\otimes H}(a\otimes b) = a\otimes b_1\otimes b_2$

とすることで ${}_H\mathcal{YD}^H$ の object となる。

$Hom_k(V\otimes H, H)\ni g\longmapsto \tilde g\in Nat(V\otimes F, F)$ を $M\in {}_H\mathcal{YD}^H$ に対して

$\tilde g_M : V\otimes M\ni v\otimes m\longmapsto g(v\otimes m_1)\triangleright m_0\in M$ として定める。

$f: M\longrightarrow N$ in ${}_H\mathcal{YD}^H$ に対して

$$\left(\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\rho_M"'] & N \arrow[d, "\rho_N"] \\
M\otimes H \arrow[r, "f\otimes id_H"'] & N\otimes H
\end{tikzcd}\right)
\qquad
\begin{tikzcd}
V\otimes M \arrow[r] \arrow[d, "\tilde g_M"'] & V\otimes N \arrow[d, "\tilde g_N"] \\
M \arrow[r] & N
\end{tikzcd}$$

$m\longmapsto f(m)\longmapsto f(m)_0\otimes f(m)_1 \overset{①}{=} f(m_0)\otimes m_1$, $\quad m\longmapsto m_0\otimes m_1\longmapsto f(m_0)\otimes m_1$

$v\otimes m\longmapsto v\otimes f(m)\longmapsto g(v\otimes f(m)_1)\triangleright f(m)_0 \overset{①}{=} g(v\otimes m_1)\triangleright f(m_0)$

$v\otimes m\longmapsto g(v\otimes m_1)\triangleright m_0\longmapsto g(v\otimes m_1)\triangleright f(m_0)$

$\therefore\ \tilde g$ は naturality をみたす。

また、 $Nat(V\otimes F, F)\ni \tau\longmapsto \check\tau = \left(v\otimes h\longmapsto (id_H\otimes\varepsilon_H)(\tau_{H\otimes H}(v\otimes 1_H\otimes h))\right)\in Hom_k(V\otimes H, H)$

$g\in Hom_k(V\otimes H, H)$ に対して

$\check{\tilde g}(v\otimes h) = (id_H\otimes\varepsilon_H)(\tilde g_{H\otimes H}(v\otimes 1_H\otimes h)) = (id_H\otimes\varepsilon_H)(g(v\otimes h_2)\triangleright(1_H\otimes h_1))$

$= (id_H\otimes\varepsilon_H)\left(g(v\otimes h_2)_2\otimes g(v\otimes h_2)_3 h_1 S^{-1}(g(v\otimes h_2)_1)\right) = \varepsilon(h_1)g(v\otimes h_2) = g(v\otimes h)\qquad \therefore\ \check{\tilde g} = g$

$M \in {}_H\mathcal{YD}^H$, $m \in M$ に対して, $-\triangleright m : H \ni h \longmapsto h\triangleright m \in M$ とすると $(-\triangleright m)(hh') = hh'\triangleright m = h\triangleright(-\triangleright m)(h')$

$\therefore$ $-\triangleright m$ は ${}_H\mathcal{M}$ の morphism $\therefore$ $(-\triangleright m)\otimes id_H : H\otimes H \longrightarrow M\otimes H$ は ${}_H\mathcal{YD}^H$ の morphism

$\therefore$ $\tau \in Nat(V\otimes F, F)$ に対して

$$\begin{tikzcd}
V\otimes H\otimes H \arrow[r, "id_V\otimes(-\triangleright m)\otimes id_H"] \arrow[d, "\tau_{H\otimes H}"'] & V\otimes M\otimes H \arrow[d, "\tau_{M\otimes H}"] \\
H\otimes H \arrow[r, "(-\triangleright m)\otimes id_H"'] & M\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
v\otimes h\otimes h' \arrow[r, mapsto] \arrow[d, mapsto] & v\otimes h\triangleright m\otimes h' \arrow[d, mapsto] \\
\tau_{H\otimes H}(v\otimes h\otimes h') \arrow[r, mapsto] & (-\triangleright m)(\tau_{H\otimes H}(v\otimes h\otimes h'))
\end{tikzcd}$$

特に $h = 1_H$ のとき

$$\tau_{M\otimes H}(v\otimes m\otimes h') = ((-\triangleright m)\otimes id_H)(\tau_{H\otimes H}(v\otimes 1_H\otimes h'))$$

また, $\rho_M : M \longrightarrow M\otimes H$ が ${}_H\mathcal{M}^H$ の morphism より

$$\begin{tikzcd}
V\otimes M \arrow[r, "id_V\otimes\rho_M"] \arrow[d, "\tau_M"'] & V\otimes M\otimes H \arrow[d, "\tau_{M\otimes H}"] \\
M \arrow[r, "\rho_M"] \arrow[dr, "id_M"'] & M\otimes H \arrow[d, "id_M\otimes\varepsilon_H"] \\
 & M
\end{tikzcd}$$

$$\begin{tikzcd}
v\otimes m \arrow[r, mapsto] \arrow[d, mapsto] & v\otimes m_0\otimes m_1 \arrow[d, mapsto] \\
 & \tau_{M\otimes H}(v\otimes m_0\otimes m_1) \arrow[d, equal] \\
\tau_M(v\otimes m) \arrow[r, mapsto] & \tau_M(v\otimes m)_0\otimes\tau_M(v\otimes m)_1
\end{tikzcd}$$

$\tilde{\check{\tau}}_M(v\otimes m) = \check{\tau}(v\otimes m_1)\triangleright m_0 = (id_H\otimes\varepsilon_H)(\tau_{H\otimes H}(v\otimes 1_H\otimes m_1))\triangleright m_0 = ((-\triangleright m_0)\circ(id_H\otimes\varepsilon_H)\circ\tau_{H\otimes H})(v\otimes 1_H\otimes m_1)$

$= ((id_H\otimes\varepsilon_H)\circ((-\triangleright m_0)\otimes id_H)\circ\tau_{H\otimes H})(v\otimes 1_H\otimes m_1) = (id_H\otimes\varepsilon_H)(\tau_{M\otimes H}(v\otimes m_0\otimes m_1)) = ((id_H\otimes\varepsilon_H)\circ\tau_{M\otimes H}\circ(id_V\otimes\Delta_M))(v\otimes m)$

$= \tau_M(v\otimes m)$ $\quad\therefore\ \tilde{\check{\tau}} = \tau$ $\quad\therefore\ Nat(V\otimes F, F) \cong Hom_k(V\otimes H, H)$

ここで, $\otimes$ と Hom は Adjoint より, $Hom_k(V\otimes H, H) \cong Hom_k(V, Hom_k(H, H))$

$\therefore$ $Nat(-\otimes F, F) \cong Hom_k(-, Hom_k(H, H))$

ここで $Hom_k(H, H) \cong Hom_k(H, Hom_k(H^*, k)) \cong Hom_k(H\otimes H^*, k) \cong (H\otimes H^*)^* \cong H\otimes H^*$

$\therefore$ $Nat(-\otimes F, F) \cong Hom_k(-, H\otimes H^*)$ ▨

## Rem

$\Theta_V : Hom_k(V, H\otimes H^*) \xrightarrow{\Phi} Hom_k(V\otimes H, H) \xrightarrow{\sim} Nat(V\otimes F, F)$

$\Theta_V^{-1} : Nat(V\otimes F, F) \xrightarrow{\vee} Hom_k(V\otimes H, H) \xrightarrow{\Phi} Hom_k(V, H\otimes H^*)$

を明示的に表すと

$f \in Hom_k(V, H\otimes H^*)$, $M \in {}_H\mathcal{M}^H$, $m \in M$, $v \in V$ に対し

$\Theta_V(f)_M(v\otimes m) = \widetilde{\Phi(f)}_M(v\otimes m) = \Phi(f)(v\otimes m_1)\triangleright m_0 = f_2(v)(m_1)f_1(v)\triangleright m_0$

$\tau \in Nat(V\otimes F, F)$, $v \in V$ とする。

$\Theta_V^{-1}(\tau)(v) = \Phi(\check{\tau})(v) = \check{\tau}(v\otimes e_i)\otimes e^i = (id_H\otimes\varepsilon_H)(\tau_{H\otimes H}(v\otimes 1_H\otimes e_i))\otimes e^i$

## Prop

上記の $H\otimes H^*$ の積構造および余積構造は以下の形で与えられる。

$$(a\bowtie\xi)\cdot(b\bowtie\nu) = ab_2\bowtie(S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3)*\nu$$

$$\Delta(a\bowtie\xi) =$$

(proof)

$M\in {}_H\mathcal{YD}^H$, $m\in M$, $a\otimes\xi\in H\otimes H^*$ をとる。

$$\varphi_M(a\otimes\xi\otimes m) = \theta_{H\otimes H^*}(id_H\otimes id_{H^*})(a\otimes\xi\otimes m) = \xi(m_1)a\triangleright m_0$$

$$(\varphi\circ(id\otimes\varphi))_M(a\otimes\xi\otimes b\otimes\nu\otimes m) = \varphi_M(a\otimes\xi\otimes\nu(m_1)b\triangleright m_0) = \nu(\underbrace{m_1}_{m_2})\,\xi(\underbrace{(b\triangleright m_0)_1}_{b_3\underbrace{(m_0)_1}_{m_1}S^{-1}(b_1)})\,a\triangleright\underbrace{(b\triangleright m_0)_0}_{b_2\triangleright\underbrace{(m_0)_0}_{m_0}}$$

$$= ((S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3)*\nu)(m_1)(ab_2)\triangleright m_0$$

$$\theta_{H\otimes H^*\otimes H\otimes H^*}^{-1}(\varphi\circ(id\otimes\varphi))(a\otimes\xi\otimes b\otimes\nu) = (id_H\otimes\varepsilon_H)((\varphi\circ(id\otimes\varphi))_{H\otimes H}(a\otimes\xi\otimes b\otimes\nu\otimes 1_H\otimes e_i))\otimes e^i$$

$$= (id_H\otimes\varepsilon_H)(((S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3)*\nu)(\underbrace{e_{i_2}}_{e_i})\underbrace{(ab_2)\triangleright(1_H\otimes e_{i_1})}_{\underbrace{a_2b_{22}}_{a\,b_2}\otimes a_3b_{23}e_{i_1}S^{-1}(a_1b_{21})})\otimes e^i = ab_2\otimes(S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3)*\nu$$

## Rem

以降上記の(Hopf)代数を $H\bowtie H^*$ と表記し、その元も $a\bowtie\xi$ などと書くことにする。

またこの積の単位元は $1_H\bowtie\varepsilon$ である。

## Prop

$H$: f.d. Hopf alg, $F: {}_H\mathcal{YD}^H \longrightarrow \mathrm{Vect}_\Bbbk$ : forgetful functor, $V$: $\Bbbk$- vector space

$\mathrm{Nat}(F, F\otimes V) \cong \mathrm{Hom}_\Bbbk(H, H\otimes V) \cong \mathrm{Hom}_\Bbbk(H^*\otimes H, V)$

(proof)

$H \in {}_H\mathcal{M}$ より、 $H\otimes H$ は $h \triangleright (a\otimes b) = h_2 a \otimes h_3 b S^{-1}(h_1)$, $\rho_{H\otimes H}(a\otimes b) = a\otimes b_1 \otimes b_2$

とすることで ${}_H\mathcal{YD}^H$ の object となる。

$\mathrm{Hom}_\Bbbk(H, H\otimes V) \ni g \longmapsto \tilde{g} \in \mathrm{Nat}(F, F\otimes V)$ を $M \in {}_H\mathcal{M}^H$ に対して

$\tilde{g}_M : M \ni m \mapsto (-\triangleright m_0 \otimes id_V)(g(m_1)) \in M\otimes V$ として定める。

$f: M \longrightarrow N$ in ${}_H\mathcal{YD}^H$ に対して

$$\left(\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"'] \arrow[dr, phantom, "① \circlearrowright"] & N \arrow[d, "\rho_N"] \\ M\otimes H \arrow[r, "f\otimes id_H"'] & N\otimes H \end{tikzcd} \qquad \begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & f(m) \arrow[d, mapsto] \\ m_0\otimes m_1 \arrow[r, mapsto] & f(m_0)\otimes m_1 = f(m)_0 \otimes f(m)_1 \end{tikzcd}\right)$$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\tilde{g}_M"'] \arrow[dr, phantom, "\circlearrowright"] & N \arrow[d, "\tilde{g}_N"] \\ M\otimes V \arrow[r, "f\otimes id_V"'] & N\otimes V \end{tikzcd} \qquad \begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & f(m) \arrow[d, mapsto] \\ (-\triangleright m_0\otimes id_V)(g(m_1)) \arrow[r, mapsto] & (-\triangleright f(m_0)\otimes id_V)(g(m_1)) = (-\triangleright f(m)_0 \otimes id_V)(g(f(m)_1)) \end{tikzcd}$$

$\therefore\ \tilde{g}$ は naturality をみたす。

また、 $\mathrm{Nat}(F, F\otimes V) \ni \tau \longmapsto \check{\tau} = \left(h \longmapsto (id_H\otimes \varepsilon_H \otimes id_V)(\tau_{H\otimes H}(1_H\otimes h))\right) \in \mathrm{Hom}_\Bbbk(H, H\otimes V)$

$g \in \mathrm{Hom}_\Bbbk(H, H\otimes V)$ に対して $\rho_{H\otimes H}(1_H\otimes h) = 1_H \otimes h_1 \otimes h_2$

$\check{\tilde{g}}(h) = (id_H\otimes \varepsilon_H \otimes id_V)\circ(\tilde{g}_{H\otimes H}(1_H\otimes h)) = (id_H\otimes\varepsilon_H\otimes id_V)(-\triangleright(1_H\otimes h_1)\otimes id_V)(\underbrace{g(h_2)}_{g_1(h_2)\otimes g_2(h_2)})$

$= (id_H\otimes \varepsilon_H\otimes id_V)(g_1(h_2)_2 \otimes g_1(h_2)_3 h_1 S^{-1}(g_1(h_2)_1) \otimes g_2(h_2)) = g_1(h)\otimes g_2(h) = g(h)$

$\therefore\ \check{\tilde{g}} = g$

$M \in {}_H\mathcal{YD}^H$, $m \in M$ に対して, $-\triangleright m : H \ni h \longmapsto h \triangleright m \in M$ は

$$\begin{tikzcd}
H\otimes H \arrow[rr, "id_H\otimes(-\triangleright m)"] \arrow[d, "\lambda_H"'] & & H\otimes M \arrow[d, "\lambda_M"] \\
H \arrow[rr, "(-\triangleright m)"'] & & M
\end{tikzcd}$$

$$\begin{tikzcd}
h\otimes a \arrow[r, mapsto] \arrow[d, mapsto] & h\otimes a\triangleright m \arrow[d, mapsto] \\
& h\triangleright(a\triangleright m) \arrow[d, phantom, "\shortparallel"] \\
ha \arrow[r, mapsto] \arrow[ur, phantom, "\circlearrowleft"] & ha\triangleright m
\end{tikzcd}$$

より $-\triangleright m$ は ${}_H\mathcal{M}$ の morphism

$\therefore\ (-\triangleright m)\otimes id_H : H\otimes H \longrightarrow M\otimes H$ は ${}_H\mathcal{YD}^H$ の morphism

$\therefore\ \tau \in Nat(F, F\otimes V)$ に対して,

$$\begin{tikzcd}
H\otimes H \arrow[rr, "(-\triangleright m)\otimes id_H"] \arrow[d, "\tau_{H\otimes H}"'] & & M\otimes H \arrow[d, "\tau_{M\otimes H}"] \\
H\otimes H\otimes V \arrow[rr, "(-\triangleright m)\otimes id_H\otimes id_V"'] & & M\otimes H\otimes V
\end{tikzcd}$$

$$\begin{tikzcd}
h\otimes h' \arrow[r, mapsto] \arrow[d, mapsto] & h\triangleright m\otimes h' \arrow[d, mapsto] \\
& \tau_{M\otimes H}(h\triangleright m\otimes h') \arrow[d, phantom, "\shortparallel"] \\
\tau_{H\otimes H}(h\otimes h') \arrow[r, mapsto] & ((-\triangleright m)\otimes id_H\otimes id_V)(\tau_{H\otimes H}(h\otimes h'))
\end{tikzcd}$$

特に $h = 1_H$ のとき,

$$((-\triangleright m)\otimes id_H\otimes id_V)(\tau_{H\otimes H}(1\otimes h')) = \tau_{M\otimes H}(m\otimes h')$$

また, $\rho_M : M \longrightarrow M\otimes H$ が ${}_H\mathcal{M}^H$ の morphism より

$$\begin{tikzcd}
M \arrow[r, "\rho_M"] \arrow[d, "\tau_M"'] & M\otimes H \arrow[d, "\tau_{M\otimes H}"] \\
M\otimes V \arrow[r, "\rho_M\otimes id_V"] \arrow[dr, "id_M\otimes id_V"'] & M\otimes H\otimes V \arrow[d, "id_M\otimes\varepsilon_H\otimes id_V"] \arrow[dl, phantom, "\circlearrowleft"] \\
& M\otimes V
\end{tikzcd}$$

$$\begin{tikzcd}
m \arrow[r, mapsto] \arrow[d, mapsto] & m_0\otimes m_1 \arrow[d, mapsto] \\
\tau_M(m) \arrow[r, mapsto] & (\rho_M\otimes id_V)(\tau_M(m)) = \tau_{M\otimes H}(m_0\otimes m_1)
\end{tikzcd}$$

$$\tilde{\check{\tau}}_M(m) = (-\triangleright m_0\otimes id_V)(\check{\tau}(m_1)) = ((-\triangleright m_0\otimes id_V)\circ(id_H\otimes\varepsilon_H\otimes id_V))(\tau_{H\otimes H}(1_H\otimes m_1))$$

$$= ((id_M\otimes\varepsilon_H\otimes id_V)\circ(-\triangleright m_0\otimes id_H\otimes id_V))(\tau_{H\otimes H}(1_H\otimes m_1)) = (id_M\otimes\varepsilon_H\otimes id_V)(\tau_{M\otimes H}(m_0\otimes m_1)) = \tau_M(m)$$

$\therefore\ \tilde{\check{\tau}} = \tau \qquad \therefore\ Nat(F, F\otimes V) \cong Hom_k(H, H\otimes V) \cong Hom_k(H^*\otimes H, V)$ ∎

<u>Rem</u>

$$\Theta_V : Hom_k(H^*\otimes H, V) \xrightarrow{\Phi'} Hom_k(H, H\otimes V) \xrightarrow{\sim} Nat(F, F\otimes V)$$

$$\Theta_V^{-1} : Nat(F, F\otimes V) \xrightarrow{\vee} Hom_k(H, H\otimes V) \xrightarrow{\Phi'} Hom_k(H^*\otimes H, V)$$

を明示的に表すと

$f \in Hom_k(H^*\otimes H, V)$, $M \in {}_H\mathcal{YD}^H$, $m \in M$ に対し

$$\Theta_V(f)_M(m) = \widetilde{\Phi(f)}_M(m) = (-\triangleright m_0\otimes id_V)(\Phi(f)(m_1)) = (-\triangleright m_0\otimes id_V)\Big(\sum_i e_i\otimes f(e^i\otimes m_1)\Big) = \sum_i e_i\triangleright m_0\otimes f(e^i\otimes m_1)$$

$\tau \in Nat(F, F\otimes V)$, $\xi\otimes a \in H^*\otimes H$ に対し

$$\Theta_V^{-1}(\tau)(\xi\otimes a) = \Phi'(\check{\tau})(\xi\otimes a) = (\xi\otimes id_V)(\check{\tau}(a)) = (\xi\otimes id_V)((id_H\otimes\varepsilon_H\otimes id_H)(\tau_{H\otimes H}(1_H\otimes a)))$$

## Prop

上記の $H^*\otimes H$ の余積構造は以下の形で与えられる。

$$\Delta(\xi\otimes a)=\sum_i \xi_2(e_{i_2})\xi_1\otimes e_{i_3}a_1 S^{-1}(e_{i_1})\otimes e^i\otimes a_2$$

(proof)

$M\in {}_H\mathcal{YD}^H$ , $m\in M$ とする。

$$\varphi_M(m)=\theta_{H^*\otimes H}(id_{H^*}\otimes id_H)_M(m)=\sum_i e_i\triangleright m_0\otimes e^i\otimes m_1$$

$$((\varphi\otimes id_{H^*\otimes H})\circ\varphi)_M(m)=(\varphi\otimes id_{H^*\otimes H})\Big(\sum_i e_i\triangleright m_0\otimes e^i\otimes m_1\Big)=\sum_{i,j} e_j\triangleright(e_i\triangleright m_0)_0\otimes e^j\otimes(e_i\triangleright m_0)_1\otimes e^i\otimes m_1$$

$$e_{i_2}\triangleright(m_0)_0 \qquad e_{i_3}(m_0)_1 S^{-1}(e_{i_1}) \qquad m_2$$

$$m_0 \qquad m_1$$

$$=\sum_{i,j} e_j e_{i_2}\triangleright m_0\otimes e^j\otimes e_{i_3}m_1 S^{-1}(e_{i_1})\otimes e^i\otimes m_2$$

$$\Delta_{H^*\otimes H}(\xi\otimes a)=\theta^{-1}_{H^*\otimes H\otimes H^*\otimes H}\big(((\varphi\otimes id_{H^*\otimes H})\circ\varphi)(\xi\otimes a)\big)=(\xi\otimes id_{H^*\otimes H\otimes H^*\otimes H})\big((id\otimes\varepsilon\otimes id)((\varphi\otimes id)\circ\varphi)_{H\otimes H}(1_H\otimes a)\big)$$

$$=(\xi\otimes id_{H^*\otimes H\otimes H^*\otimes H})(id_H\otimes\varepsilon_H\otimes id_{H^*\otimes H\otimes H^*\otimes H})\Big(\sum_{i,j} e_j e_{i_2}\triangleright(1_H\otimes a_1)\otimes e^j\otimes e_{i_3}a_2 S^{-1}(e_{i_1})\otimes e^i\otimes a_3\Big)=\sum_{i,j}\xi(e_j e_{i_2})e^j\otimes e_{i_3}a_1 S^{-1}(e_{i_1})\otimes e^i\otimes a_2$$

$$e_{j_2}e_{i_{22}}\otimes e_{j_3}e_{i_{23}}a_1 S^{-1}(e_{j_2}e_{i_{21}}) \qquad a_1 \qquad a_2 \qquad \xi_1(e_j)\xi_2(e_{i_2})\ \xi_1$$

$$e_j e_{i_2}$$

$$=\sum_i \xi_2(e_{i_2})\xi_1\otimes e_{i_3}a_1 S^{-1}(e_{i_1})\otimes e^i\otimes a_2$$

## Rem

$F:{}_H\mathcal{YD}^H\longrightarrow \mathrm{Vect}_{\mathbb{k}}$ : forgetful functor に対して $A_F\cong(C_F)^*$ である。

$$a\otimes\xi,\ b\otimes v\in(C_F)^*=(H^*\otimes H)^*=H\otimes H^*$$

$$\big(\mu_{(C_F)^*}(a\otimes\xi\otimes b\otimes v)\big)(\zeta\otimes c)=\big((a\otimes\xi\otimes b\otimes v)\circ\Delta_{C_F}\big)(\zeta\otimes c)=(a\otimes\xi\otimes b\otimes v)\Big(\sum_i\zeta_2(e_{i_2})\zeta_1\otimes e_{i_3}c_1 S^{-1}(e_{i_1})\otimes e^i\otimes c_2\Big)$$

$$=\sum_i\zeta_2(e_{i_2})\zeta_1(a)\xi(e_{i_3}c_1 S^{-1}(e_{i_1}))e^i(b)v(c_2)=\zeta_1(a)\xi_2(c_1)v(c_2)\xi_3(S^{-1}(b_1))\zeta_2(b_2)\xi_1(b_3)=(ab_2\otimes(S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3)*v)(\zeta\otimes c)$$

$$\xi_1(e_{i_3})\xi_2(c_1)\xi_3(S^{-1}(e_{i_1})) \qquad \zeta(ab_2)\ (\xi_2*v)(c)$$

$$((S^{-1})^*(\xi_3)*\zeta_2*\xi_1)(e_i) \qquad S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3$$

$$\therefore\ \mu_{(C_F)^*}(a\otimes\xi\otimes b\otimes v)=ab_2\otimes(S^{-1}(b_1)\rightharpoonup\xi\leftharpoonup b_3)*v=\mu_{A_F}((a\bowtie\xi)\otimes(b\bowtie v))$$

## Prop

$H$: f.d. Hopf alg, $\triangleleft : {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H \longrightarrow {}_H\mathcal{M}^H$ を $((M,\lambda_M,\rho_M),(N,\lambda_N,\rho_N)) \in {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$ に対して

$M \triangleleft N = M \otimes N$, $\lambda_{M\triangleleft N}(h\otimes m\otimes n) = h_1 \triangleright m \otimes h_2 \triangleright n$, $\rho_{M\triangleleft N}(m\otimes n) = m_0 \otimes n_0 \otimes n_1 m_1$ とする。

このとき ${}_H\mathcal{M}^H$ は right ${}_H\mathcal{YD}^H$-module category

(proof)

まず $\triangleleft$ が functor であることを確かめる。$\lambda_{M\triangleleft N}$ により $M\triangleleft N$ が left $H$-module になることは trivial

$$\begin{tikzcd}
M\triangleleft N \arrow[r, "\rho_{M\triangleleft N}"] \arrow[d, "\rho_{M\triangleleft N}"'] & M\triangleleft N\otimes H \arrow[d, "id_{M\triangleleft N}\otimes\Delta_H"] \\
M\triangleleft N\otimes H \arrow[r, "\rho_{M\triangleleft N}\otimes id_H"'] & M\triangleleft N\otimes H\otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] & m_0\otimes n_0\otimes n_1 m_1 \arrow[d, mapsto] \\
m_0\otimes n_0\otimes n_1 m_1 \arrow[r, mapsto] & m_0\otimes n_0\otimes n_1 m_1\otimes n_2 m_2
\end{tikzcd}$$

$$\begin{tikzcd}
M\triangleleft N\otimes H \arrow[r, "id_{M\triangleleft N}\otimes\varepsilon_H"] & M\triangleleft N\otimes k \arrow[dl, bend left] \\
M\triangleleft N \arrow[u, "\rho_{M\triangleleft N}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
m_0\otimes n_0\otimes n_1 m_1 \arrow[r, mapsto] & m\otimes n\otimes 1_k \arrow[dl, mapsto] \\
m\otimes n \arrow[u, mapsto] &
\end{tikzcd}$$

$\therefore\ (M\triangleleft N, \rho_{M\triangleleft N}) \in \mathcal{M}^H$　　また $h\in H$, $m\otimes n \in M\triangleleft N$ に対して

$(h\triangleright(m\otimes n))_0 \otimes (h\triangleright(m\otimes n))_1 = (h_1\triangleright m\otimes h_2\triangleright n)_0 \otimes (h_1\triangleright m\otimes h_2\triangleright n)_1 = (h_1\triangleright m)_0 \otimes (h_2\triangleright n)_0 \otimes (h_2\triangleright n)_1 (h_1\triangleright m)_1$

$= h_1\triangleright m_0 \otimes (h_3\triangleright n)_0 \otimes (h_3\triangleright n)_1 (h_2 m_1) = h_1\triangleright m_0 \otimes (h_3\triangleright n)_0 \otimes (h_3\triangleright n)_1 h_2 m_1 = h_1\triangleright m_0 \otimes h_2\triangleright n_0 \otimes h_3 n_1 m_1$

$\|$

$h_1\triangleright(m\otimes n)_0 \otimes h_2(m\otimes n)_1 = h_1\triangleright(m_0\otimes n_0)\otimes h_2 n_1 m_1 = h_1\triangleright m_0 \otimes h_2\triangleright n_0 \otimes h_3 n_1 m_1$

$\therefore\ M\triangleleft N \in {}_H\mathcal{M}^H$　　また $(f,g) : (M,N) \longrightarrow (M',N')$ in ${}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$ に対して

$f\triangleleft g : M\triangleleft N \ni m\otimes n \longmapsto f(m)\otimes g(n) \in M'\triangleleft N'$ は

$$\begin{tikzcd}
H\otimes M\triangleleft N \arrow[r, "id_H\otimes(f\triangleleft g)"] \arrow[d, "\lambda_{M\triangleleft N}"'] & H\otimes M'\triangleleft N' \arrow[d, "\lambda_{M'\triangleleft N'}"] \\
M\triangleleft N \arrow[r, "f\triangleleft g"'] & M'\triangleleft N'
\end{tikzcd}
\qquad
\begin{tikzcd}
h\otimes m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] & h\otimes f(m)\otimes g(n) \arrow[d, mapsto] \\
h_1\triangleright m\otimes h_2\triangleright n \arrow[r, mapsto] & h_1\triangleright f(m)\otimes h_2\triangleright g(n)
\end{tikzcd}$$

$$\begin{tikzcd}
M\triangleleft N \arrow[r, "f\triangleleft g"] \arrow[d, "\rho_{M\triangleleft N}"'] & M'\triangleleft N' \arrow[d, "\rho_{M'\triangleleft N'}"] \\
M\triangleleft N\otimes H \arrow[r, "f\triangleleft g\otimes id_H"'] & M'\triangleleft N'\otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] & f(m)\otimes g(n) \arrow[d, mapsto] \\
m_0\otimes n_0\otimes n_1 m_1 \arrow[r, mapsto] & f(m_0)\otimes g(n_0)\otimes n_1 m_1 = f(m)_0\otimes g(n)_0\otimes g(n)_1 f(m)_1
\end{tikzcd}$$

$\therefore\ f\triangleleft g$ は ${}_H\mathcal{M}^H$ の morphism

$(M,N) \xrightarrow{(f,g)} (M',N') \xrightarrow{(f',g')} (M'',N'')$ in ${}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$ に対して

$\triangleleft((f',g')\circ(f,g)) = \triangleleft(f'\circ f, g'\circ g) = (f'\circ f)\triangleleft(g'\circ g) = (f'\circ f)\otimes(g'\circ g) = (f'\otimes g')\circ(f\otimes g) = \triangleleft(f',g')\circ\triangleleft(f,g)$

$\triangleleft((id_M, id_N)) = id_M\otimes id_N = id_{M\otimes N} = id_{M\triangleleft N}$　　$\therefore\ \triangleleft$ は functor

$(M,N,L) \in {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H \times {}_H\mathcal{YD}^H$ に対して $(M\triangleleft N)\triangleleft L$ は

$\Delta_{(M\triangleleft N)\triangleleft L}(m\otimes n\otimes l) = (m\otimes n)_0 \otimes l_0 \otimes l_1 (m\otimes n)_1 = m_0\otimes n_0\otimes l_0\otimes l_1 n_1 m_1$

$\Delta_{M\triangleleft(N\otimes L)}(m\otimes n\otimes l) = m_0\otimes(n\otimes l)_0\otimes(n\otimes l)_1 m_1 = m_0\otimes n_0\otimes l_0\otimes l_1 n_1 m_1$

$\therefore\ id_{M\otimes N\otimes L} : (M\triangleleft N)\triangleleft L = (M\otimes N)\otimes L \longrightarrow M\otimes(N\otimes L) = M\triangleleft(N\otimes L)$ は natural iso

で pentagon axiom, triangle axiom を満たし ${}_H\mathcal{M}^H$ は right ${}_H\mathcal{YD}^H$-module category となる。■

Prop

$H$: f.d. Hopf alg, $G_1 : {}_H\mathcal{M}^H \longrightarrow \mathrm{Vect}_k$, $G_2 : {}_H\mathcal{YD}^H \longrightarrow \mathrm{Vect}_k$ : forgetful functor

$V$ : $k$-vector space, $G_1\otimes G_2 : {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H \longrightarrow \mathrm{Vect}_k$

$\mathrm{Nat}(V\otimes G_1\otimes G_2,\ G_1\otimes G_2) \cong \mathrm{Hom}_k(V\otimes H\otimes H,\ H\otimes H) \cong \mathrm{Hom}_k(V,\ \mathrm{End}_k(H\otimes H))$

(proof)

$H\in{}_H\mathcal{M}$ より、$H\otimes H$ は $h\triangleright(a\otimes b) = h_1 a\otimes h_2 b$, $\rho_{H\otimes H}(a\otimes b) = a\otimes b_1\otimes b_2$

とすることで ${}_H\mathcal{M}^H$ の object となる。

$H\in{}_H\mathcal{M}$ より、$H\otimes H$ は $h\triangleright(a\otimes b) = h_2 a\otimes h_3 b S(h_1)$, $\rho_{H\otimes H}(a\otimes b) = a\otimes b_1\otimes b_2$

とすることで ${}_H\mathcal{YD}^H$ の object となる。

$\mathrm{Hom}_k(V\otimes H\otimes H,\ H\otimes H) \ni \varphi \longmapsto \tilde{\varphi}\in\mathrm{Nat}(V\otimes G_1\otimes G_2,\ G_1\otimes G_2)$ を $(M,N)\in{}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$

に対して、$\tilde{\varphi}_{(M,N)} : V\otimes M\otimes N \ni v\otimes m\otimes n \longmapsto \varphi(v\otimes m_1\otimes n_1)\triangleright(m_0\otimes n_0)\in M\otimes N$ と定める。

$(f,g) : (M,N)\longrightarrow(M',N')$ in ${}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$ に対して

$$\begin{tikzcd}
V\otimes M\otimes N \arrow[r, "id_V\otimes f\otimes g"] \arrow[d, "\tilde{\varphi}_{(M,N)}"'] & V\otimes M'\otimes N' \arrow[d, "\tilde{\varphi}_{(M',N')}"] \\
M\otimes N \arrow[r, "f\otimes g"'] & M'\otimes N'
\end{tikzcd}$$

$$\begin{tikzcd}
v\otimes m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & v\otimes f(m)\otimes g(n) \arrow[d, mapsto] \\
\varphi(v\otimes m_1\otimes n_1)\triangleright(m_0\otimes n_0) \arrow[r, mapsto] & \varphi(v\otimes \underset{m_1}{f(m)_1}\otimes \underset{n_1}{g(n)_1})\triangleright(\underset{f(m_0)}{f(m)_0}\otimes \underset{g(n_0)}{g(n)_0})
\end{tikzcd}$$

$\therefore\ \tilde{\varphi}$ は Naturality をみたす。

また、$\mathrm{Nat}(V\otimes G_1\otimes G_2,\ G_1\otimes G_2)\ni\tau \longmapsto \check{\tau}\in\mathrm{Hom}_k(V\otimes H\otimes H,\ H\otimes H)$

$\check{\tau} = \left(v\otimes h\otimes h' \longmapsto ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H))(\tau_{H\otimes H,\,H\otimes H}(v\otimes 1_H\otimes h\otimes 1_H\otimes h'))\right)$ とする。

$\varphi \in \mathrm{Hom}_{\Bbbk}(V\otimes H\otimes H,\ H\otimes H)$

$\check{\tilde{\varphi}}(v\otimes h\otimes h') = ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H))(\tilde{\varphi}_{H\otimes H,H\otimes H}(v\otimes 1_H\otimes h\otimes 1_H\otimes h'))$

$= ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H))\,\varphi(v\otimes h_2\otimes h_2')\triangleright(1_H\otimes h_1\otimes 1_H\otimes h_1') = \varphi(v\otimes h\otimes h')\triangleright(1_H\otimes 1_H) = \varphi(v\otimes h\otimes h')$

$(M,N)\in {}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$, $m\in M$, $n\in N$ に対して, $(-\triangleright m,\ -\triangleright n):(H,H)\longrightarrow(M,N)$ は ${}_H\mathcal{M}\times{}_H\mathcal{M}$ の morphism より.

$((-\triangleright m)\otimes id_H,\ (-\triangleright n)\otimes id_H):(H\otimes H,\ H\otimes H)\longrightarrow(M\otimes H,\ N\otimes H)$ in ${}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$

$\tau\in\mathrm{Nat}(V\otimes G_1\otimes G_2,\ G_1\otimes G_2)$ に対して,

$$\begin{tikzcd}
V\otimes(H\otimes H)\otimes(H\otimes H) \arrow[r, "id_V\otimes(-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H"] \arrow[d, "\tau_{H\otimes H,H\otimes H}"'] & V\otimes(M\otimes H)\otimes(N\otimes H) \arrow[d, "\tau_{M\otimes H,N\otimes H}"] \\
(H\otimes H)\otimes(H\otimes H) \arrow[r, "(-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H"'] & (M\otimes H)\otimes(N\otimes H)
\end{tikzcd}$$

$$\begin{tikzcd}
v\otimes a\otimes a'\otimes b\otimes b' \arrow[r, mapsto] \arrow[d, mapsto] & v\otimes(a\triangleright m)\otimes a'\otimes b\triangleright n\otimes b' \arrow[d, mapsto] \\
\tau_{H\otimes H,H\otimes H}(v\otimes a\otimes a'\otimes b\otimes b') \arrow[r, mapsto] & ((-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H)(\tau_{H\otimes H,H\otimes H}(v\otimes a\otimes a'\otimes b\otimes b'))
\end{tikzcd}$$

特に $a=b=1_H$ のとき

$\tau_{M\otimes H,N\otimes H}(v\otimes m\otimes a'\otimes n\otimes b') = \tau_{M\otimes H,N\otimes H}(v\otimes(1_H\triangleright m)\otimes a'\otimes(1_H\triangleright n)\otimes b')$

$= ((-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H)(\tau_{H\otimes H,H\otimes H}(v\otimes 1_H\otimes a'\otimes 1_H\otimes b'))$

$(\Delta_M,\Delta_N):(M,N)\longrightarrow(M\otimes H,\ N\otimes H)$ は ${}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$ の morphism より.

$$\begin{tikzcd}
V\otimes M\otimes N \arrow[r, "id_V\otimes\rho_M\otimes\rho_N"] \arrow[d, "\tau_{M,N}"'] & V\otimes M\otimes H\otimes N\otimes H \arrow[d, "\tau_{M\otimes H,N\otimes H}"] \\
M\otimes N \arrow[r, "\rho_M\otimes\rho_N"'] \arrow[dr, "id_{M\otimes N}"'] & M\otimes H\otimes N\otimes H \arrow[d, "id_M\otimes\varepsilon_H\otimes id_N\otimes\varepsilon_H"] \\
& M\otimes N
\end{tikzcd}$$

$$\begin{tikzcd}
v\otimes m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] & v\otimes m_0\otimes m_1\otimes n_0\otimes n_1 \arrow[d, mapsto] \\
\tau_{M,N}(v\otimes m\otimes n) \arrow[r, mapsto] & (\rho_M\otimes\rho_N)(\tau_{M,N}(v\otimes m\otimes n)) = \tau_{M\otimes H,N\otimes H}(v\otimes m_0\otimes m_1\otimes n_0\otimes n_1)
\end{tikzcd}$$

$\check{\tilde{\tau}}_{M,N}(v\otimes m\otimes n) = \check{\tau}_{M,N}(v\otimes m_1\otimes n_1)\triangleright(m_0\otimes n_0)$

$= ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H))(\tau_{H\otimes H,H\otimes H}(v\otimes 1_H\otimes m_1\otimes 1_H\otimes n_1))\triangleright(m_0\otimes n_0)$

$= ((-\triangleright m_0)\circ(id_H\otimes\varepsilon_H)\otimes(-\triangleright n_0)\circ(id_H\otimes\varepsilon_H))(\tau_{H\otimes H,H\otimes H}(v\otimes 1_H\otimes m_1\otimes 1_H\otimes n_1))$

$= ((id_H\otimes\varepsilon_H)\circ((-\triangleright m_0)\otimes id_H)\otimes(id_H\otimes\varepsilon_H)\circ((-\triangleright n_0)\otimes id_H))(\tau_{H\otimes H,H\otimes H}(v\otimes 1_H\otimes m_1\otimes 1_H\otimes n_1))$

$= ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H))(\tau_{M\otimes H,N\otimes H}(v\otimes m_0\otimes m_1\otimes n_0\otimes n_1)) = \tau_{M,N}(v\otimes m\otimes n)$ $\quad\therefore\ \check{\tilde{\tau}} = \tau$

$\therefore\ \mathrm{Nat}(V\otimes G_1\otimes G_2,\ G_1\otimes G_2)\cong\mathrm{Hom}_{\Bbbk}(V\otimes H\otimes H,\ H\otimes H)\cong\mathrm{Hom}_{\Bbbk}(V,\ \mathrm{Hom}_{\Bbbk}(H\otimes H,\ H\otimes H))\cong\mathrm{Hom}_{\Bbbk}(V,\ \mathrm{End}(H\otimes H))$ ∎

## Rem

$f \in \mathrm{Hom}_{\Bbbk}(V, H\otimes H^*\otimes H\otimes H^*)$, $v \in V$, $a\otimes b \in H\otimes H$ に対して, $\Phi(f)(v\otimes a\otimes b) = f(v)_2(a)\, f(v)_4(b)\, f(v)_1 \otimes f(v)_3$

$g \in \mathrm{Hom}_{\Bbbk}(V\otimes H\otimes H, H\otimes H)$, $v\in V$ に対して, $\Psi(g)(v) = g(v\otimes e_i\otimes e_j)_1 \otimes e^i \otimes g(v\otimes e_i\otimes e_j)_2 \otimes e^j$ とする。

$$(\Phi\circ\Psi)(g)(v\otimes a\otimes b) = \Phi(\Psi(g))(v\otimes a\otimes b) = e^i(a)e^j(b)\, g(v\otimes e_i\otimes e_j) = g(v\otimes a\otimes b)$$

$$(\Psi\circ\Phi)(f)(v) = \Psi(\Phi(f))(v) = \Phi(f)(v\otimes e_i\otimes e_j)_1 \otimes e^i \otimes \Phi(f)(v\otimes e_i\otimes e_j)_2 \otimes e^j = f(v)_2(e_i)\, f(v)_4(e_j)\, f(v)_1 \otimes \underbrace{e^i}_{f(v)_2} \otimes f(v)_3 \otimes \underbrace{e^j}_{f(v)_4} = f(v)$$

$$\theta_V : \mathrm{Hom}_{\Bbbk}(V, H\otimes H^*\otimes H\otimes H^*) \xrightarrow{\Phi} \mathrm{Hom}_{\Bbbk}(V\otimes H\otimes H, H\otimes H) \xrightarrow{\sim} \mathrm{Nat}(V\otimes G_1\otimes G_2, G_1\otimes G_2)$$

$\theta_V^{-1} : \mathrm{Nat}(V\otimes G_1\otimes G_2, G_1\otimes G_2) \xrightarrow{\vee} \mathrm{Hom}_{\Bbbk}(V\otimes H\otimes H, H\otimes H) \xrightarrow{\Psi} \mathrm{Hom}_{\Bbbk}(V, H\otimes H^*\otimes H\otimes H^*)$ を明示的に表すと

$f \in \mathrm{Hom}_{\Bbbk}(V, H\otimes H^*\otimes H\otimes H^*)$, $M\triangleleft N \in {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$, $m\otimes n \in M\triangleleft N$, $v\in V$ に対し

$$(\theta_V(f))_{M\triangleleft N}(v\otimes m\otimes n) = (\widetilde{\Phi(f)})_{M\triangleleft N}(v\otimes m\otimes n) = \Phi(f)(v\otimes m_1\otimes n_1) \triangleright (m_0\otimes n_0) = f(v)_2(m_1)\, f(v)_4(n_1)(f(v)_1\otimes f(v)_3)\triangleright(m_0\otimes n_0)$$

$= f(v)_2(m_1)\, f(v)_4(n_1)\, f(v)_1\triangleright m_0 \otimes f(v)_3 \triangleright n_0$ 　また, $\tau \in \mathrm{Nat}(V\otimes G_1\otimes G_2, G_1\otimes G_2)$, $v\in V$ に対して,

$$\theta_V^{-1}(\tau)(v) = \Psi(\check{\tau})(v) = \check{\tau}(v\otimes e_i\otimes e_j)_1 \otimes e^i \otimes \check{\tau}(v\otimes e_i\otimes e_j)_2 \otimes e^j$$

$$= ((\mathrm{id}_H\otimes\varepsilon_H)\otimes(\mathrm{id}_H\otimes\varepsilon_H))(\tau_{H\otimes H, H\otimes H}(v\otimes 1_H\otimes e_i\otimes 1_H\otimes e_j))_1 \otimes e^i \otimes ((\mathrm{id}_H\otimes\varepsilon_H)\otimes(\mathrm{id}_H\otimes\varepsilon_H))(\tau_{H\otimes H, H\otimes H}(v\otimes 1_H\otimes e_i\otimes 1_H\otimes e_j))_2 \otimes e^j$$

## Prop

$H\otimes H^*\otimes H\otimes H^*$ の積構造は以下の形で与えられる。

$$((a\# \xi)\otimes(a'\bowtie\xi'))\cdot((b\# \nu)\otimes(b'\bowtie\nu')) = ab_1 \# (\xi \leftharpoonup b_2)*\nu \otimes a'b'_2 \bowtie (S^{-1}(b'_1)\rightharpoonup \xi' \leftharpoonup b'_3)*\nu'$$

(proof)

$(M,N) \in {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$, $m\in M$, $n\in N$, $a\otimes\xi\otimes a'\otimes\xi' \in H\otimes H^*\otimes H\otimes H^*$

$$\varphi_{M\triangleleft N}(a\otimes\xi\otimes a'\otimes\xi'\otimes m\otimes n) = \theta_{H\otimes H^*\otimes H\otimes H^*}(\mathrm{id}_H\otimes\mathrm{id}_{H^*}\otimes\mathrm{id}_H\otimes\mathrm{id}_{H^*})(a\otimes\xi\otimes a'\otimes\xi'\otimes m\otimes n) = \xi(m_1)\, a\triangleright m_0 \otimes \xi'(n_1)\, a'\triangleright n_0$$

$$(\varphi\circ(\mathrm{id}\otimes\varphi))_{M\triangleleft N}(a\otimes\xi\otimes a'\otimes\xi'\otimes b\otimes\nu\otimes b'\otimes\nu'\otimes m\otimes n) = \varphi_{M\triangleleft N}(a\otimes\xi\otimes a'\otimes\xi'\otimes \nu(m_1)\, b\triangleright m_0 \otimes \nu'(n_1)\, b'\triangleright n_0)$$

$$= \nu(\underbrace{m_1}_{m_2})\,\nu'(\underbrace{n_1}_{n_2})\,\xi(\underbrace{(b\triangleright m_0)_1}_{b_2 (m_0)_1 \to m_1})\, a\triangleright(\underbrace{(b\triangleright m_0)_0}_{b_1\triangleright (m_0)_0 \to m_0}) \otimes \xi'(\underbrace{(b'\triangleright n_0)_1}_{b'_3 (n_0)_1 S^{-1}(b'_1) \to n_1})\, a'\triangleright(\underbrace{(b'\triangleright n_0)_0}_{b'_2\triangleright (n_0)_0 \to n_0}) = \nu(m_2)\nu'(n_2)\,\xi(b_2 m_1)(ab_1)\triangleright m_0 \otimes \xi'(b'_3 n_1 S^{-1}(b'_1))(a'b'_2)\triangleright n_0$$

$$(\mathrm{id}_H\otimes\varepsilon_H\otimes\mathrm{id}_H\otimes\varepsilon_H)((\varphi\circ(\mathrm{id}\otimes\varphi))_{H\otimes H^*\triangleleft H\otimes H^*}(a\otimes\xi\otimes a'\otimes\xi'\otimes b\otimes\nu\otimes b'\otimes\nu'\otimes 1_H\otimes e_i\otimes 1_H\otimes e_j)) = \xi(b_2 e_{i_1})\nu(e_{i_2})$$

$$\nu(\underbrace{e_{i_3}}_{e_{i_2}})\,\nu'(\underbrace{e_{j_3}}_{e_{j_2}})\,\xi(b_2\underbrace{e_{i_2}}_{e_{i_1}})(ab_1)\triangleright(1_H\otimes e_{i_1}) \otimes \xi'(b'_3\underbrace{e_{j_2}}_{e_{j_1}} S^{-1}(b'_1))(a'b'_2)\triangleright(1_H\otimes e_{j_1})$$

$$\underbrace{a_1b_{11}}_{ab_1}\otimes a_2b_{12}e_{i_1} \qquad \underbrace{a'_2 b'_{22}}_{a'b'_2}\otimes a'_3 b'_{23} e_{j_1} S^{-1}(a'_1 b'_{21})$$

$$= ((\xi\leftharpoonup b_2)*\nu)(e_i)\,((S^{-1}(b'_1)\rightharpoonup\xi'\leftharpoonup b'_3)*\nu')(e_j)\; ab_1 \otimes a'b'_2$$

$$\theta^{-1}_{H\otimes H^*\otimes H\otimes H^*\otimes H\otimes H^*\otimes H\otimes H^*}(\varphi\circ(\mathrm{id}\otimes\varphi))(a\otimes\xi\otimes a'\otimes\xi'\otimes b\otimes\nu\otimes b'\otimes\nu') = ((\xi\leftharpoonup b_2)*\nu)(e_i)\,((S^{-1}(b'_1)\rightharpoonup\xi'\leftharpoonup b'_3)*\nu')(e_j)\; ab_1\otimes e^i\otimes a'b'_2\otimes e^j$$

$$= ab_1 \otimes (\xi\leftharpoonup b_2)*\nu \otimes a'b'_2 \otimes (S^{-1}(b'_1)\rightharpoonup\xi'\leftharpoonup b'_3) \qquad \blacksquare$$

## Prop

$$\begin{tikzcd}[column sep=large]
{}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H \arrow[rr, "\triangleleft"] \arrow[dr, "G_1 \otimes G_2"'] & \arrow[d, phantom, "\circlearrowright"] & {}_H\mathcal{M}^H \arrow[dl, "G_1"] \\
& \mathrm{Vect}_{\Bbbk} &
\end{tikzcd}$$

により、$\begin{array}{ccc} \mathrm{End}(G_1) & \longrightarrow & \mathrm{End}(G_1 \times G_2) \\ \wr\| & & \wr\| \\ \mathcal{H}(H) & & \mathcal{H}(H) \otimes D(H^*) \end{array}$ が定まり、

$\mathcal{H}(H^*)$ は right - $D(H^*)$ comodule algebra で

$$\Delta_{\mathcal{H}(H)} : \mathcal{H}(H) \ni a \# \xi \longmapsto a_1 \# \xi_2 \otimes a_2 \bowtie \xi_1 \in \mathcal{H}(H) \otimes D(H^*)$$

(proof)

$$\chi : \mathrm{End}(G_1) \ni \tau \longmapsto \Phi(\tau) \in \mathrm{End}(G_1 \otimes G_2) \qquad \chi(\tau)_{M,N} = \tau_{M \triangleright N}$$

$$\begin{array}{ccccccc}
\mathcal{H}(H) & \xrightarrow{\theta_{\Bbbk}} & \mathrm{End}(G_1) & \xrightarrow{\chi} & \mathrm{End}(G_1 \otimes G_2) & \xrightarrow{\theta_{\Bbbk}^{-1}} & \mathcal{H}(H) \otimes D(H^*) \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
a \# \xi & \longmapsto & \theta_{\Bbbk}(a \# \xi) & \longmapsto & \chi(\theta_{\Bbbk}(a \# \xi)) & \longmapsto & \theta_{\Bbbk}^{-1}(\chi(\theta_{\Bbbk}(a \# \xi)))
\end{array}$$

$$\theta_{\Bbbk}^{-1}(\chi(\theta_{\Bbbk}(a \# \xi))) = ((id_H \otimes \varepsilon_H) \otimes (id_H \otimes \varepsilon_{\Bbbk}))(\chi(\theta_{\Bbbk}(a \# \xi))_{H \otimes H \otimes H \otimes H}(1_H \otimes e_i \otimes 1_H \otimes e_j)) \otimes e^i \otimes e^j$$

$$= ((id_H \otimes \varepsilon_H) \otimes (id_H \otimes \varepsilon_{\Bbbk}))\, \theta_{\Bbbk}(a \# \xi)_{(H \otimes H) \triangleleft (H \otimes H)}((1_H \otimes e_i) \otimes (1_H \otimes e_j)) \otimes e^i \otimes e^j$$

$$= ((id_H \otimes \varepsilon_H) \otimes (id_H \otimes \varepsilon_{\Bbbk}))\Big(\xi(e_{j_2} e_{i_2})\, a \triangleright (1_H \otimes e_{i_1} \otimes 1_H \otimes e_{j_1})\Big) \otimes e^i \otimes e^j$$

$$= \xi(e_{j_2} e_{i_2})((id_H \otimes \varepsilon_H) \otimes (id_H \otimes \varepsilon_{\Bbbk}))(\underbrace{a_1 \triangleright (1_H \otimes e_{i_1})}_{a_1 \otimes a_2 e_{i_1}} \otimes \underbrace{a_2 \triangleright (1_H \otimes e_{j_1})}_{a_4 \otimes a_5 e_{j_1} S^{-1}(a_3)}) \otimes e^i \otimes e^j$$

$$= \xi(e_j e_i)\, a_1 \otimes a_2 \otimes e^i \otimes e^j = a_1 \# \xi_2 \otimes a_2 \bowtie \xi_1$$

## Prop

$H$ : f.d. Hopf alg, $\quad G_1 : {}_H\mathcal{M}^H \longrightarrow \mathrm{Vect}_{\Bbbk}$, $\quad G_2 : {}_H\mathcal{YD}^H \longrightarrow \mathrm{Vect}_{\Bbbk}$ : forgetful functor

$\mathbb{V}$ : $\Bbbk$-vector space, $\quad G_1 \otimes G_2 : {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H \longrightarrow \mathrm{Vect}_{\Bbbk}$

$$\mathrm{Nat}(G_1\otimes G_2,\ G_1\otimes G_2\otimes \mathbb{V}) \cong \mathrm{Hom}_{\Bbbk}(H\otimes H,\ H\otimes H\otimes \mathbb{V}) \cong \mathrm{Hom}_{\Bbbk}((H\otimes H)^*\otimes (H\otimes H),\ \mathbb{V})$$

(proof)

$H \in {}_H\mathcal{M}$ より、$H\otimes H$ は $h \triangleright (a\otimes b) = h_1 a \otimes h_2 b$, $\quad \rho_{H\otimes H}(a\otimes b) = a\otimes b_1 \otimes b_2$

とすることで ${}_H\mathcal{M}^H$ の object となる。

$H \in {}_H\mathcal{M}$ より、$H\otimes H$ は $h \triangleright (a\otimes b) = h_2 a \otimes h_3 b\, S(h_1)$, $\quad \rho_{H\otimes H}(a\otimes b) = a\otimes b_1 \otimes b_2$

とすることで ${}_H\mathcal{YD}^H$ の object となる。

$\mathrm{Hom}_{\Bbbk}(H\otimes H,\ H\otimes H\otimes \mathbb{V}) \ni \varphi \longmapsto \tilde{\varphi} \in \mathrm{Nat}(G_1\otimes G_2,\ G_1\otimes G_2\otimes \mathbb{V})$ を $(M, N) \in {}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$

に対して、$\tilde{\varphi}_{(M,N)} : M\otimes N \ni m\otimes n \longmapsto ((-\triangleright m_0)\otimes(-\triangleright n_0)\otimes id_{\mathbb{V}})(\varphi(m_1\otimes n_1)) \in M\otimes N\otimes \mathbb{V}$

$(f, g) : (M, N) \longrightarrow (M', N')$ in ${}_H\mathcal{M}^H \times {}_H\mathcal{YD}^H$ に対して

$$\begin{tikzcd}
M\otimes N \arrow[rrr, "f\otimes g"] \arrow[ddd, "\tilde{\varphi}_{(M,N)}"'] & & & M'\otimes N' \arrow[ddd, "\tilde{\varphi}_{(M',N')}"] \\
& m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & f(m)\otimes g(n) \arrow[d, mapsto] & \\
& ((-\triangleright m_0)\otimes(-\triangleright n_0)\otimes id_{\mathbb{V}})(\varphi(m_1\otimes n_1)) \arrow[r, mapsto] & ((-\triangleright \underset{f(m)_0}{f(m_0)})\otimes(-\triangleright \underset{g(n)_0}{g(n_0)})\otimes id_{\mathbb{V}})(\varphi(\underset{f(m)_1}{m_1}\otimes \underset{g(n)_1}{n_1})) & \\
M\otimes N\otimes \mathbb{V} \arrow[rrr, "f\otimes g\otimes id_{\mathbb{V}}"'] & & & M'\otimes N'\otimes \mathbb{V}
\end{tikzcd}$$

$\therefore$ $\tilde{\varphi}$ は Naturality をみたす。

また、$\mathrm{Nat}(G_1\otimes G_2,\ G_1\otimes G_2\otimes \mathbb{V}) \ni \tau \longmapsto \check{\tau} \in \mathrm{Hom}_{\Bbbk}(H\otimes H,\ H\otimes H\otimes \mathbb{V})$ を

$\check{\tau}(h\otimes h') = ((id_H\otimes \varepsilon_H)\otimes(id_H\otimes\varepsilon_H)\otimes id_{\mathbb{V}})(\tau_{H\otimes H,\, H\otimes H}(1_H\otimes h\otimes 1_H\otimes h'))$ とする。

$\varphi \in \mathrm{Hom}_{\Bbbk}(H\otimes H,\ H\otimes H\otimes \mathbb{V})$ に対して、

$$\check{\tilde{\varphi}}(h\otimes h') = ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H)\otimes id_{\mathbb{V}})(\tilde{\varphi}_{H\otimes H,H\otimes H}(1_H\otimes h\otimes 1_H\otimes h'))$$

$$= ((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H)\otimes id_{\mathbb{V}})((-\triangleright(1_H\otimes h_{(1)}))\otimes(-\triangleright(1_H\otimes h'_{(1)}))\otimes id_{\mathbb{V}})(\varphi(h_{(2)}\otimes h'_{(2)})) = \varphi(h\otimes h')$$

$(M,N)\in {}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$, $m\in M$, $n\in N$ に対して、$(-\triangleright m, -\triangleright n):(H,H)\longrightarrow(M,N)$ は ${}_H\mathcal{M}\times{}_H\mathcal{M}$ の morphism より、

$$((-\triangleright m)\otimes id_H,\ (-\triangleright n)\otimes id_H):(H\otimes H,\ H\otimes H)\longrightarrow(M\otimes H,\ N\otimes H) \quad \text{in } {}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$$

$\tau\in Nat(G_1\otimes G_2,\ G_1\otimes G_2\otimes\mathbb{V})$ に対して、

$$\begin{tikzcd}
(H\otimes H)\otimes(H\otimes H) \arrow[rrr, "(-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H"] \arrow[ddd, "\tau_{H\otimes H,H\otimes H}"'] & & & (M\otimes H)\otimes(N\otimes H) \arrow[ddd, "\tau_{M\otimes H,N\otimes H}"] \\
& a\otimes a'\otimes b\otimes b' \arrow[r, mapsto] \arrow[d, mapsto] & a\triangleright m\otimes a'\otimes b\triangleright n\otimes b' \arrow[d, mapsto] & \\
& \tau_{H\otimes H,H\otimes H}(a\otimes a'\otimes b\otimes b') \arrow[r, mapsto] & ((-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H\otimes id_{\mathbb{V}})(\tau_{H\otimes H,H\otimes H}(a\otimes a'\otimes b\otimes b')) & \\
(H\otimes H)\otimes(H\otimes H)\otimes\mathbb{V} \arrow[rrr, "(-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H\otimes id_{\mathbb{V}}"'] & & & (M\otimes H)\otimes(N\otimes H)\otimes\mathbb{V}
\end{tikzcd}$$

特に $a=b=1_H$ のとき

$$\tau_{M\otimes H,N\otimes H}(m\otimes a'\otimes n\otimes b') = \tau_{M\otimes H,N\otimes H}((1_H\triangleright m)\otimes a'\otimes(1_H\triangleright n)\otimes b')$$

$$= ((-\triangleright m)\otimes id_H\otimes(-\triangleright n)\otimes id_H\otimes id_{\mathbb{V}})(\tau_{H\otimes H,H\otimes H}(1_H\otimes a'\otimes 1_H\otimes b'))$$

$(\rho_M,\rho_N):(M,N)\longrightarrow(M\otimes H,\ N\otimes H)$ は ${}_H\mathcal{M}^H\times{}_H\mathcal{YD}^H$ の morphism より、

$$\begin{tikzcd}
M\otimes N \arrow[rrr, "\rho_M\otimes\rho_N"] \arrow[ddd, "\tau_{M,N}"'] & & & M\otimes H\otimes N\otimes H \arrow[ddd, "\tau_{M\otimes H,N\otimes H}"] \\
& m\otimes n \arrow[r, mapsto] \arrow[d, mapsto] & m_0\otimes m_1\otimes n_0\otimes n_1 \arrow[d, mapsto] & \\
& \tau_{M,N}(m\otimes n) \arrow[r, mapsto] & \tau_{M\otimes H,N\otimes H}(m_0\otimes m_1\otimes n_0\otimes n_1) = (\rho_M\otimes\rho_N\otimes id_{\mathbb{V}})(\tau_{M,N}(m\otimes n)) & \\
M\otimes N\otimes\mathbb{V} \arrow[rrr, "\rho_M\otimes\rho_N\otimes id_{\mathbb{V}}"'] \arrow[drrr, "id_{M\otimes N}"'] & & & M\otimes H\otimes N\otimes H\otimes\mathbb{V} \arrow[d, "id_M\otimes\varepsilon_H\otimes id_N\otimes\varepsilon_H"] \\
& & & M\otimes N
\end{tikzcd}$$

$$\tilde{\check{\tau}}_{M,N}(m\otimes n) = ((-\triangleright m_0)\otimes(-\triangleright n_0)\otimes id_{\mathbb{V}})(\check{\tau}(m_1\otimes n_1))$$

$$= ((-\triangleright m_0)\otimes(-\triangleright n_0)\otimes id_{\mathbb{V}})\big(((id_H\otimes\varepsilon_H)\otimes(id_H\otimes\varepsilon_H)\otimes id_{\mathbb{V}})(\tau_{H\otimes H,H\otimes H}(1_H\otimes m_1\otimes 1_H\otimes n_1))\big)$$

$$= ((id_M\otimes\varepsilon_H)\otimes(id_N\otimes\varepsilon_H)\otimes id_{\mathbb{V}})\circ((-\triangleright m_0)\otimes id_H\otimes(-\triangleright n_0)\otimes id_H\otimes id_{\mathbb{V}})(\tau_{H\otimes H,H\otimes H}(1_H\otimes m_1\otimes 1_H\otimes n_1))$$

$$= ((id_M\otimes\varepsilon_H)\otimes(id_N\otimes\varepsilon_H)\otimes id_{\mathbb{V}})(\tau_{M\otimes H,N\otimes H}(m_0\otimes m_1\otimes n_0\otimes n_1)) = \tau_{M,N}(m\otimes n)$$

$\therefore\ \tilde{\check{\tau}} = \tau$

$\therefore\ Nat(G_1\otimes G_2,\ G_1\otimes G_2\otimes\mathbb{V}) \cong \mathrm{Hom}_{\Bbbk}(H\otimes H,\ H\otimes H\otimes\mathbb{V}) \cong \mathrm{Hom}_{\Bbbk}((H\otimes H)^*\otimes(H\otimes H),\ \mathbb{V})$ ∎

## Prop

上記の $(H\otimes H)^*\otimes(H\otimes H)$ の余積構造は以下の形で与えられる。

$$\Delta(\xi\otimes a\otimes \nu\otimes b) = (\xi_1\otimes e_j a_1 S^{-1}(e_i))\otimes(e^i * \xi_2 * e^j\otimes a_2)\otimes \nu_1\otimes e_k b_1\otimes \nu_2 * e^k\otimes b_2$$

(proof)

$$\theta : \mathrm{Hom}_{\mathbb{k}}((H\otimes H)^*\otimes(H\otimes H), \mathbb{V}) \longrightarrow \mathrm{Nat}(G_1\otimes G_2, G_1\otimes G_2\otimes \mathbb{V})$$

$$\theta_{\mathbb{k}} : \mathrm{Hom}_{\mathbb{k}}((H\otimes H)^*\otimes(H\otimes H), \mathbb{k}) \cong ((H\otimes H)^*\otimes(H\otimes H))^* \ni a\#\xi\otimes a'\bowtie\xi' \mapsto \theta_{\mathbb{k}}(a\#\xi\otimes a'\bowtie\xi') \in \mathrm{Nat}(G_1\otimes G_2, G_1\otimes G_2\otimes\mathbb{k})$$

$$\theta_{\mathbb{k}}(a\#\xi\otimes a'\bowtie\xi')_{M,N}(m,n) = \xi(m_1)\, a\triangleright m_0\otimes \xi'(n_1)\, a'\triangleright n_0 \in M\otimes N$$

$$\theta_{\mathbb{k}}^{-1} : \mathrm{Nat}(G_1\otimes G_2, G_1\otimes G_2\otimes\mathbb{k}) \ni \tau \longmapsto \theta_{\mathbb{k}}^{-1}(\tau) \in ((H\otimes H)^*\otimes(H\otimes H))^*$$

$$\theta_{\mathbb{k}}^{-1}(\tau)(\xi\otimes a\otimes\xi'\otimes a') = (\xi\otimes\varepsilon_H\otimes\xi'\otimes\varepsilon_H)(\tau_{H\otimes H, H\otimes H}(1_H\otimes a\otimes 1_H\otimes a'))$$

$$\theta_{\mathbb{k}}^{-1}(\theta_{\mathbb{k}}(a\#\xi\otimes a'\bowtie\xi')\circ\theta_{\mathbb{k}}(b\#\nu\otimes b'\bowtie\nu'))(\zeta\otimes c\otimes\zeta'\otimes c')$$

$$= (\zeta\otimes\varepsilon_H\otimes\zeta'\otimes\varepsilon_H)(\theta_{\mathbb{k}}(a\#\xi\otimes a'\bowtie\xi')\circ\theta_{\mathbb{k}}(b\#\nu\otimes b'\bowtie\nu'))_{H\otimes H, H\otimes H}(1_H\otimes c\otimes 1_H\otimes c'))$$

$$= ((\zeta\otimes\varepsilon_H\otimes\zeta'\otimes\varepsilon_H)\circ(\theta_{\mathbb{k}}(a\#\xi\otimes a'\bowtie\xi')_{H\otimes H, H\otimes H}))(\nu(c_2)\,\underbrace{b\triangleright(1_H\otimes c_1)}_{b_1\otimes b_2 c_1}\otimes\nu'(c'_2)\,\underbrace{b'\triangleright(1_H\otimes c'_1)}_{b'_2\otimes b'_3 c'_1 S^{-1}(b'_1)})$$

$$= \nu(c_3)\nu'(c'_3)(\zeta\otimes\varepsilon_H\otimes\zeta'\otimes\varepsilon_H)(\xi(b_3 c_2)\,\underbrace{a\triangleright(b_1\otimes b_2 c_1)}_{a_1 b_1\otimes a_2 b_2 c_1}\otimes\xi'(b'_5 c'_2 S^{-1}(b'_1))\,\underbrace{a'\triangleright(b'_3\otimes b'_4 c'_1 S^{-1}(b'_2))}_{a'_2 b'_3\otimes a'_3 b'_4 c'_1 S^{-1}(b'_2) S^{-1}(a'_1)})$$

$$= \nu(c_2)\xi(b_2 c_1)\nu'(c'_2)\xi'(b'_3 c'_1 S^{-1}(b'_1))\zeta(ab_1)\zeta'(a'b'_2)$$

$$= ((ab_1\#(\xi\leftharpoonup b_2)*\nu)\otimes(a'b'_2\bowtie(S^{-1}(b'_1)\rightharpoonup\xi'\leftharpoonup b'_3)*\nu'))(\zeta\otimes c\otimes\zeta'\otimes c')$$

$\therefore$ $(H\otimes H)^*\otimes(H\otimes H)$ の余積構造は

$$\Delta(\xi\otimes a\otimes \nu\otimes b) = (\xi_1\otimes e_j a_1 S^{-1}(e_i))\otimes(e^i * \xi_2 * e^j\otimes a_2)\otimes \nu_1\otimes e_k b_1\otimes \nu_2 * e^k\otimes b_2 \quad ▨$$

## Prop

$$\begin{tikzcd} {}_H M^H \times {}_H \mathcal{YD}^H \arrow[rr, "\triangleleft"] \arrow[dr, "G_1\otimes G_2"'] & \circlearrowright & {}_H M^H \arrow[dl, "G_1"] \\ & \mathrm{Vect}_{\mathbb{k}} & \end{tikzcd}$$

により、$\mathrm{Coend}(G_1\otimes G_2) \longrightarrow \mathrm{Coend}(G_1)$ が定まり、

$$\begin{array}{ccc} \mathrm{Coend}(G_1\otimes G_2) & \longrightarrow & \mathrm{Coend}(G_1) \\ \wr\wr & & \wr\wr \\ \mathcal{H}(H)^*\otimes D(H)^* & & \mathcal{H}(H)^* \end{array}$$

$\mathcal{H}(H)^*$ は left $D(H^*)^*$-module coalgebra

(proof)

End の場合と全く同様に示せる。▨