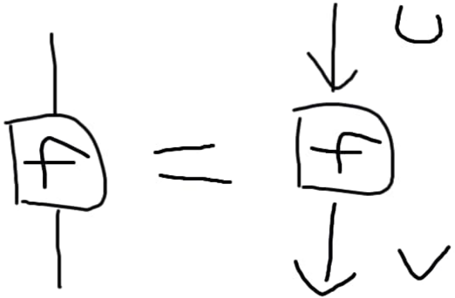
Hopf $\mathbb{k}$-alg & graphical notation. 1-xn2

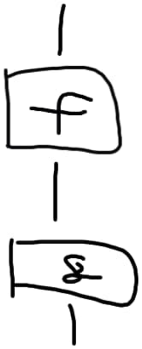
U, V, W, X, Y etc $\mathbb{k}$ - Vect sp's

$U \xrightarrow{f} V$, $W \xrightarrow{g} X$ etc linear maps

Notations

$U \xrightarrow{f} V =$ 

$U \xrightarrow{f} V \xrightarrow{g} W$ =



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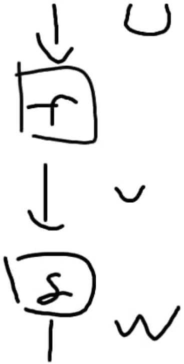
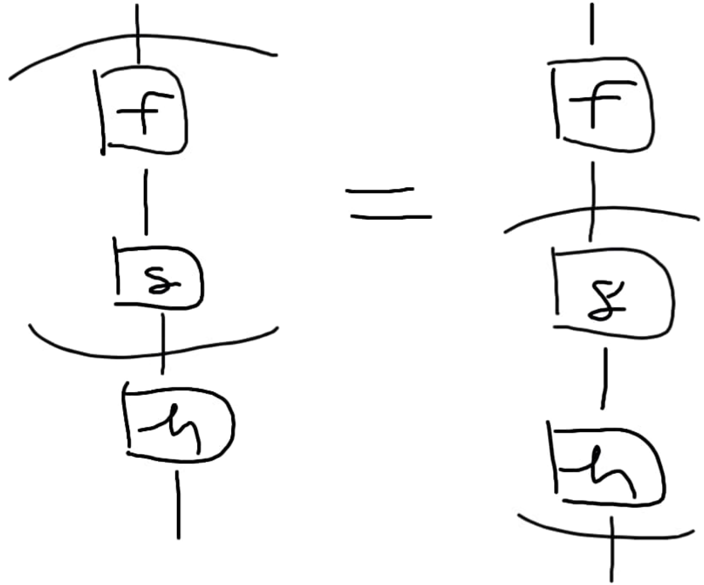


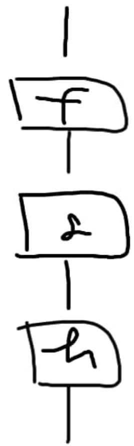
$U \xrightarrow{id_U} U$ =

$x \longmapsto x$

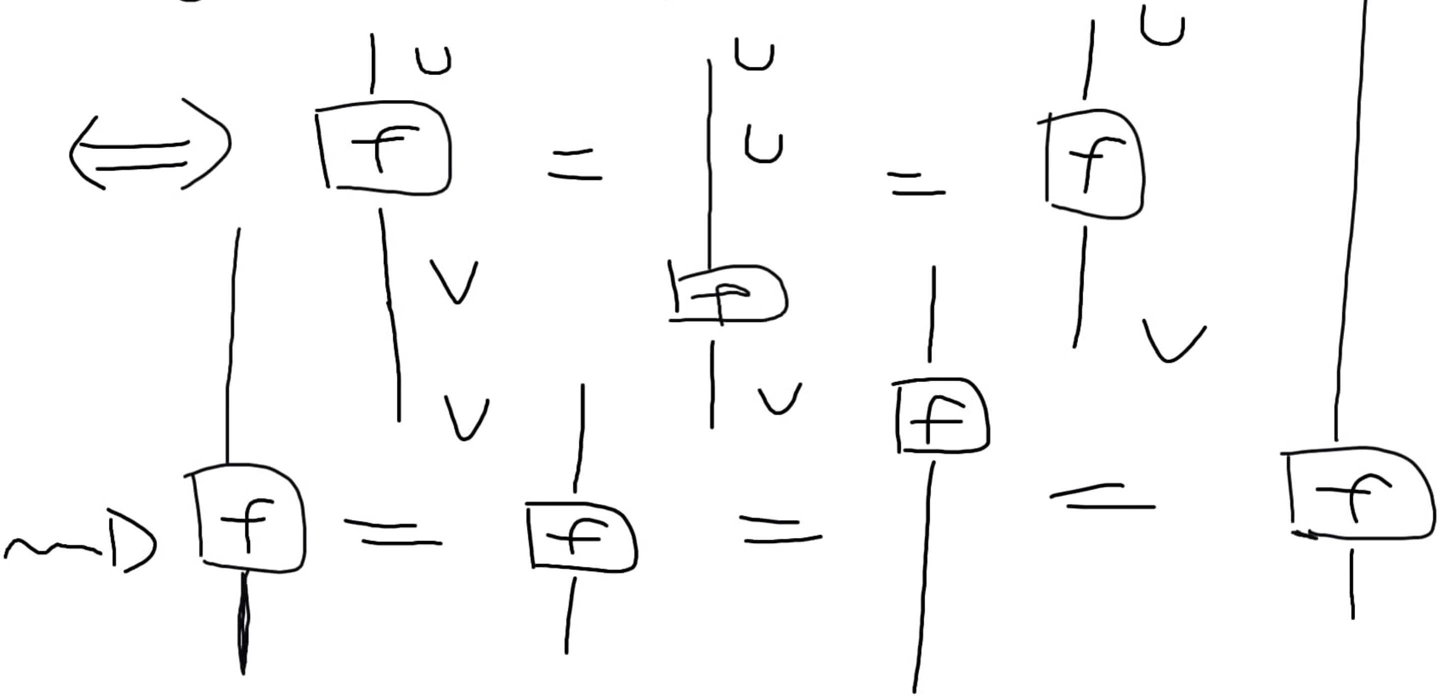
=



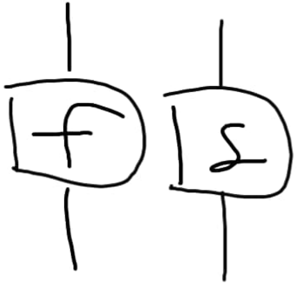
associativity.

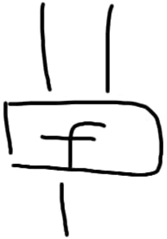
→ 中を単に  とかいてOK.

$$\mathrm{id} \circ f = f \circ \mathrm{id} = f$$



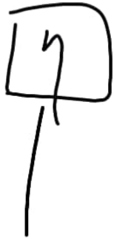
$U \xrightarrow{f} W,\ V \xrightarrow{g} X \rightsquigarrow U \otimes V \xrightarrow{f \otimes g} W \otimes X$

$=$ 

$U \otimes W \xrightarrow{f} X$ $=$  , $U \xrightarrow{g} V \otimes W$ $=$ 

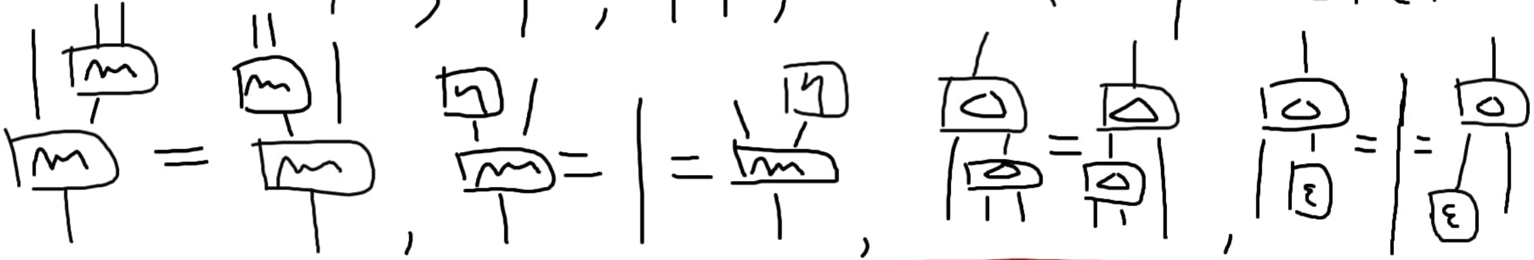
$U \otimes \mathbb{1} \simeq U$ =  かかない

$\mathbb{1} \otimes U \simeq U$ = 

$U \xrightarrow{\varepsilon} \mathbb{1}$ =  , $\mathbb{1} \xrightarrow{\eta} U$ = 

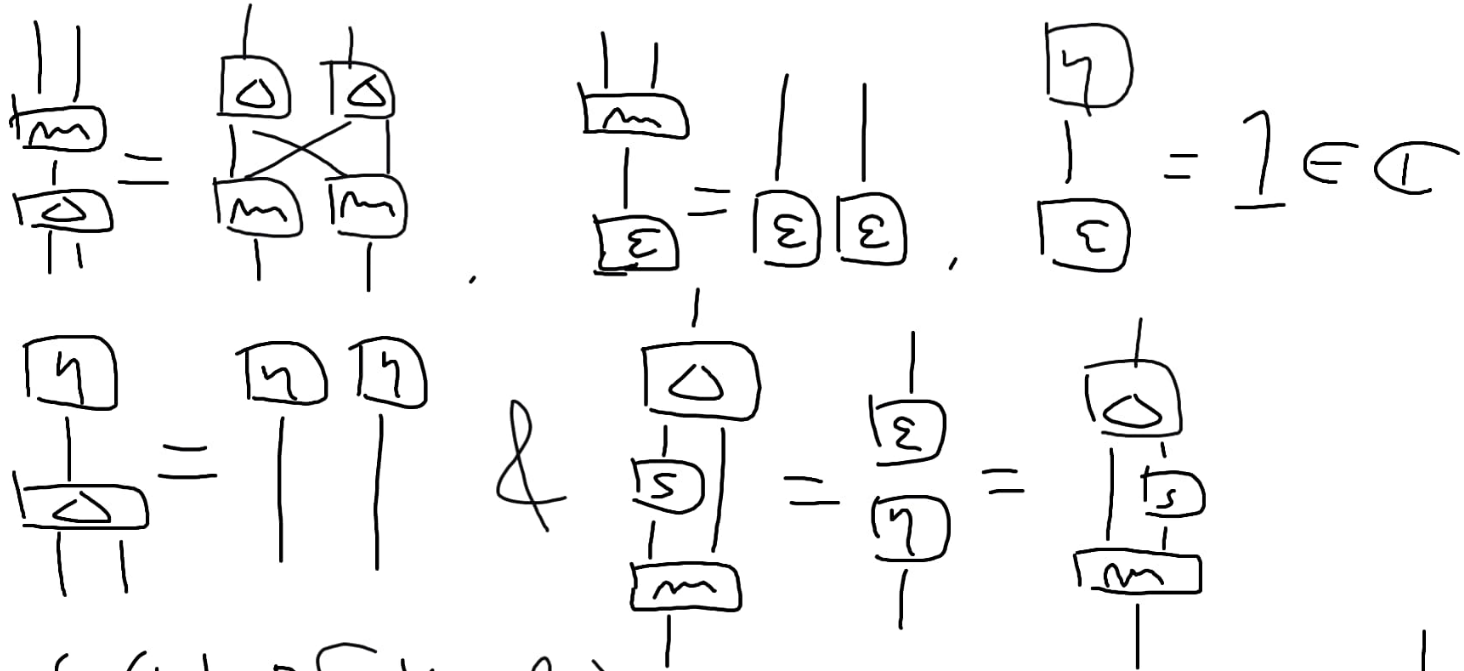
Def (Hopf alg) A : $\mathbb{C}$-vect sp is a **Hopf alg** if A is equipped with the linear maps

s.t.



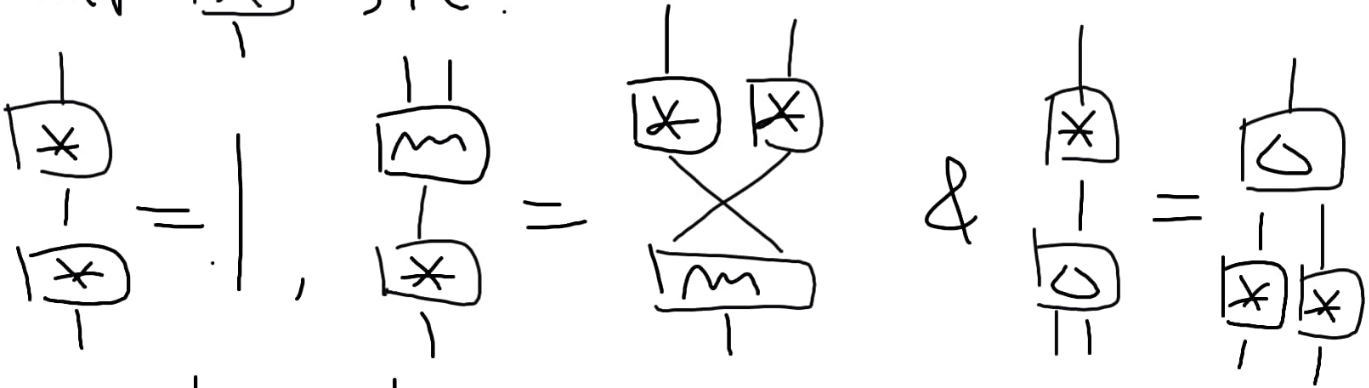
associativity

coassociativity

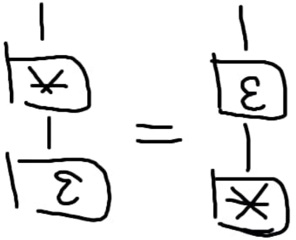


Def (Hopf $*$-alg) A : Hopf $*$-alg

def
$\Longleftrightarrow$ A : Hopf alg equipped with a conj lin map $\boxed{*}$ s.t.



Rem



は性質として証明される。

例

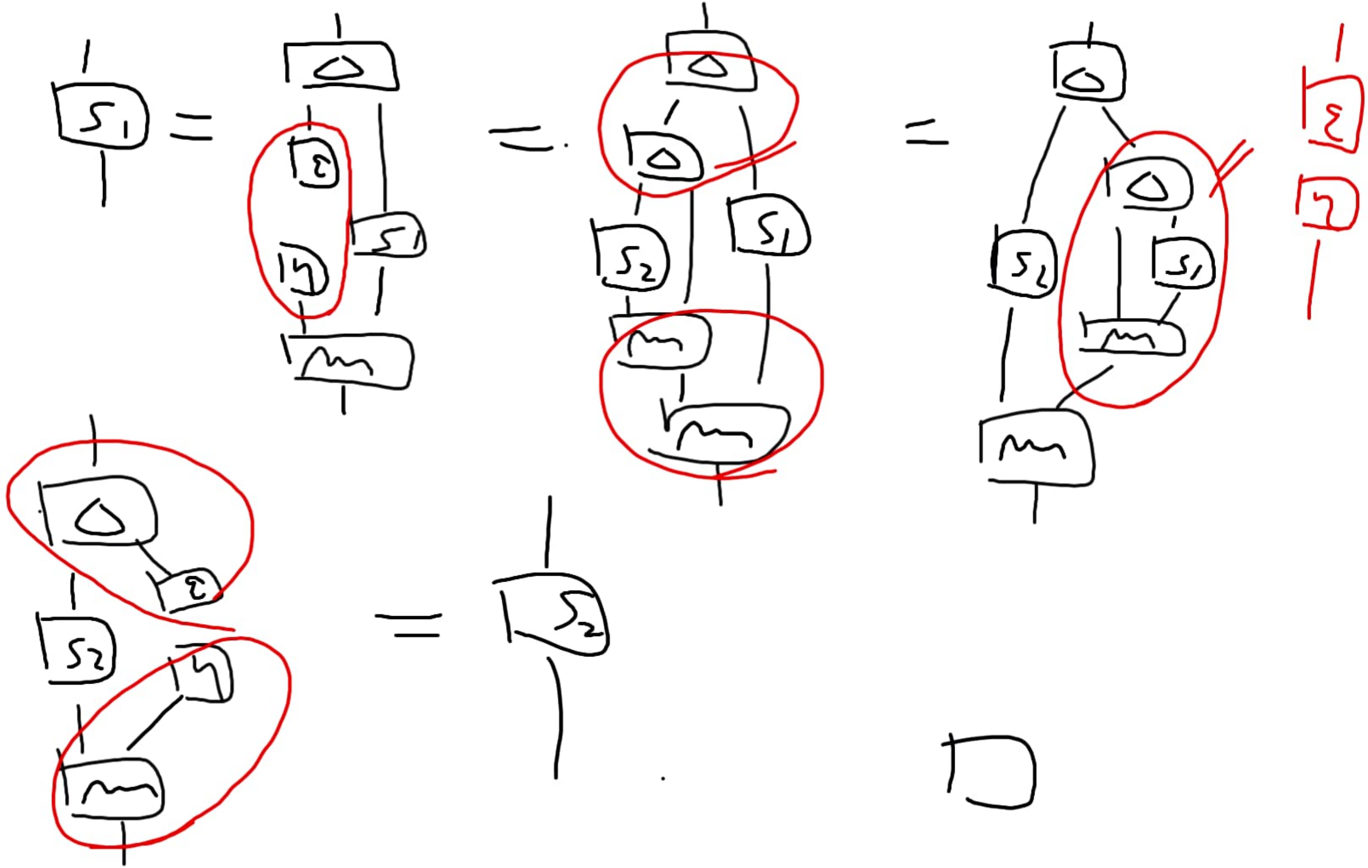
- $U(\mathfrak{sl}_n(\mathbb{C}))$
- $\mathbb{C}[G]$ (G : fin grp)
- $C(G)$ (")

Prop 1 A : Hopf $(*\text{-})$ alg

S_1, S_2 : antipodes of A

$\Longrightarrow S_1 = S_2$

Pf

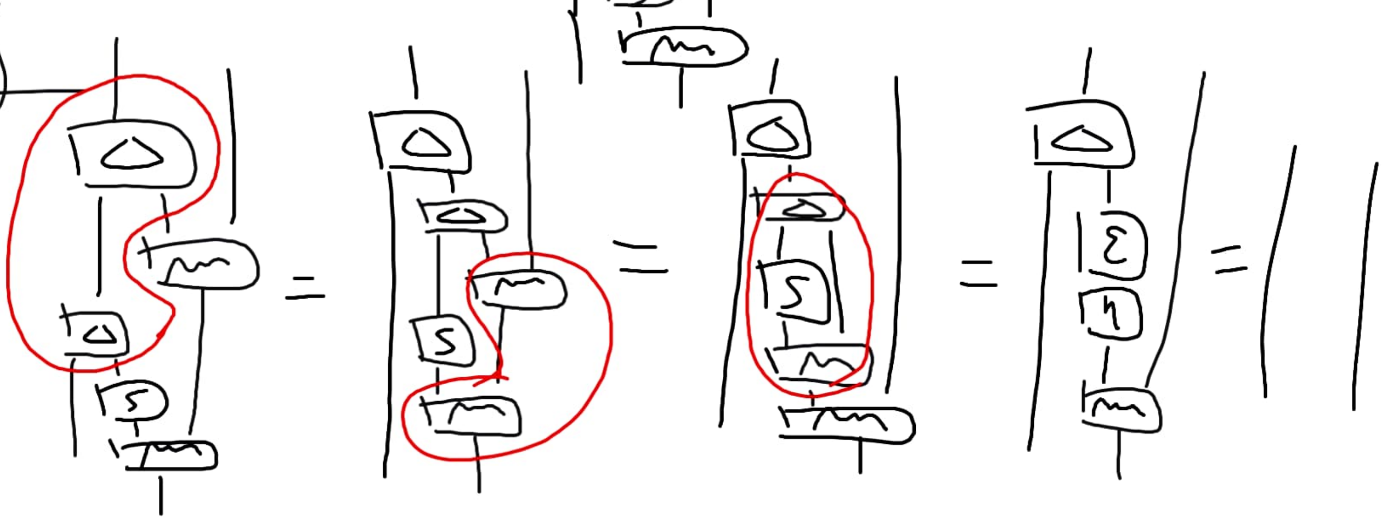


□

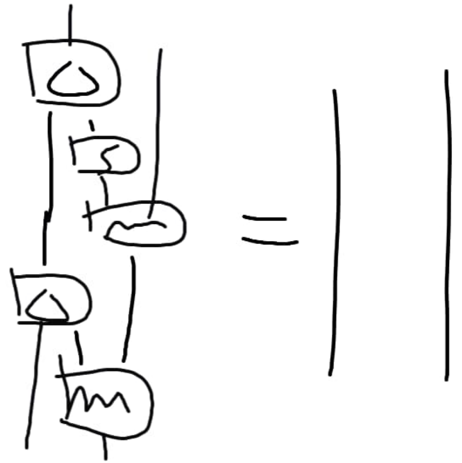
Step 2 The "ladder"  is invertible

with the inv 

Pf



同様に、



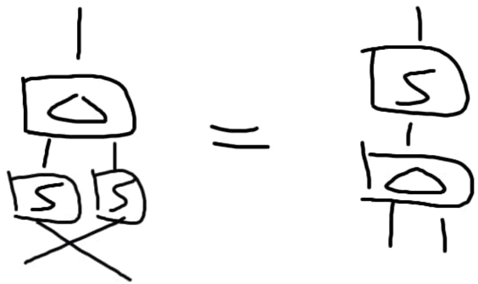
も分かる。 □

・すべての Hopf alg の S が inv とは限らない。(cf. "Free Hopf algs gen. by coalgs" by Mitsuhiro TAKEUCHI)

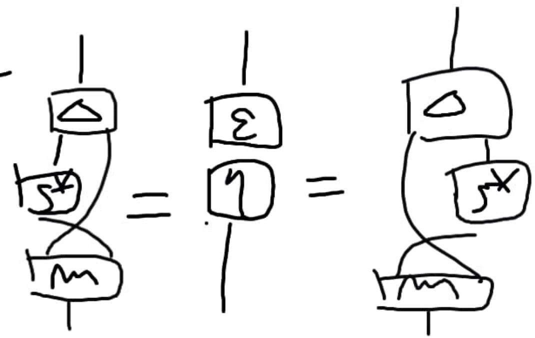
Thm3 Hopf $*$-alg の S は invertible. $S^{-1} = * \circ S \circ *$

($\rightsquigarrow$ Fact $(A, \Delta)^{op}$, $(A, \Delta)^{cop}$; Hopf alg)

lem4.



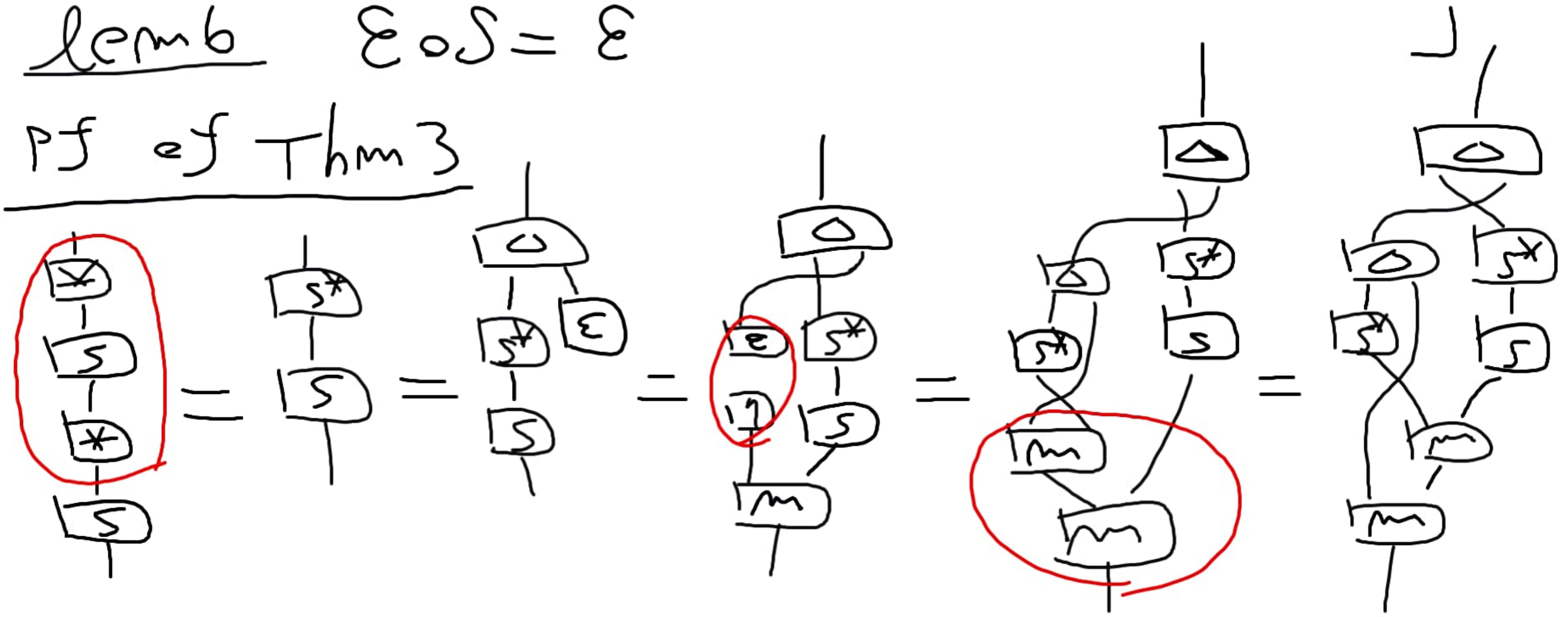
lem5. On Hopf $*$-alg's

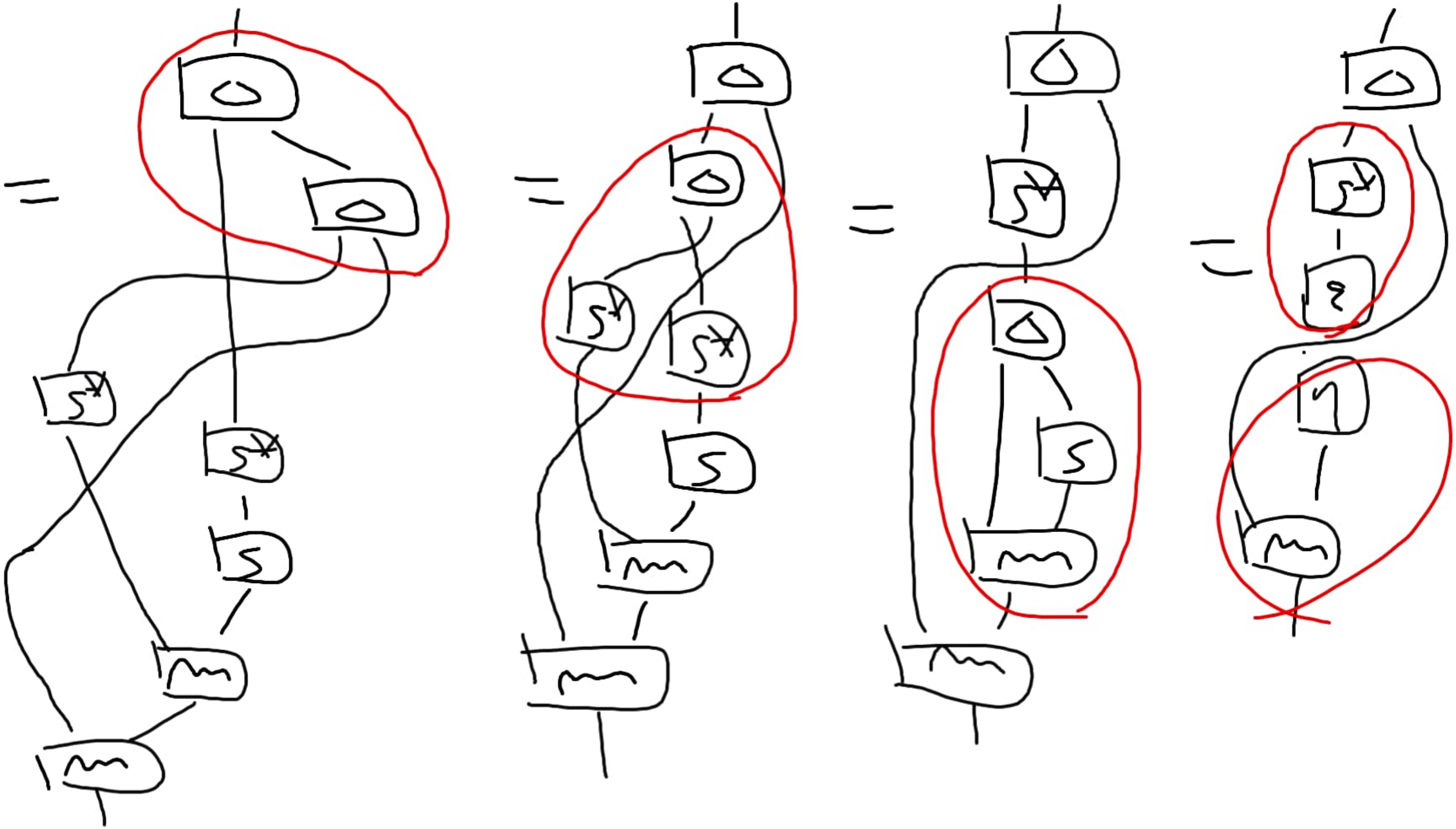
$S^* :=$  $\rightsquigarrow$ 

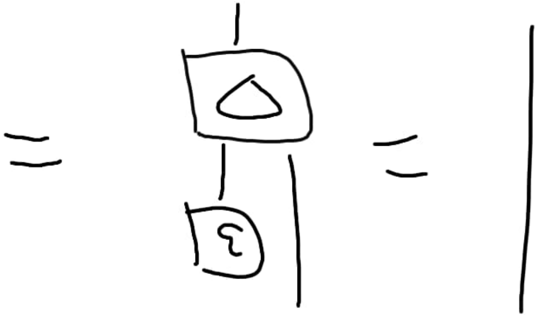
(i.e. S^* is the antipode of $(A.\Delta)^{op}$)

lemb $\varepsilon \circ S = \varepsilon$

pf of Thm 3

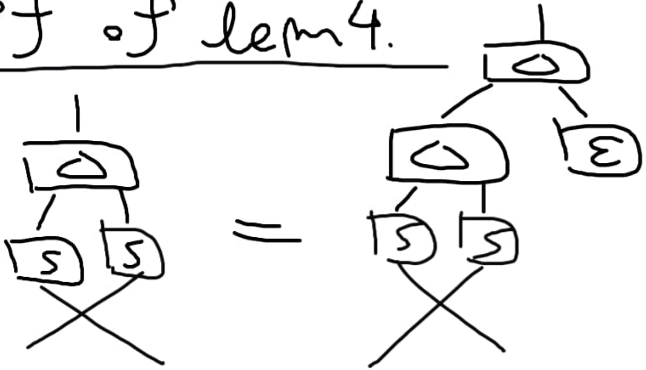


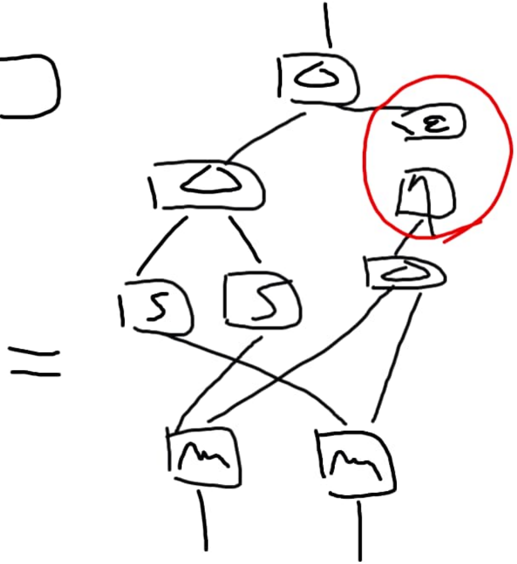


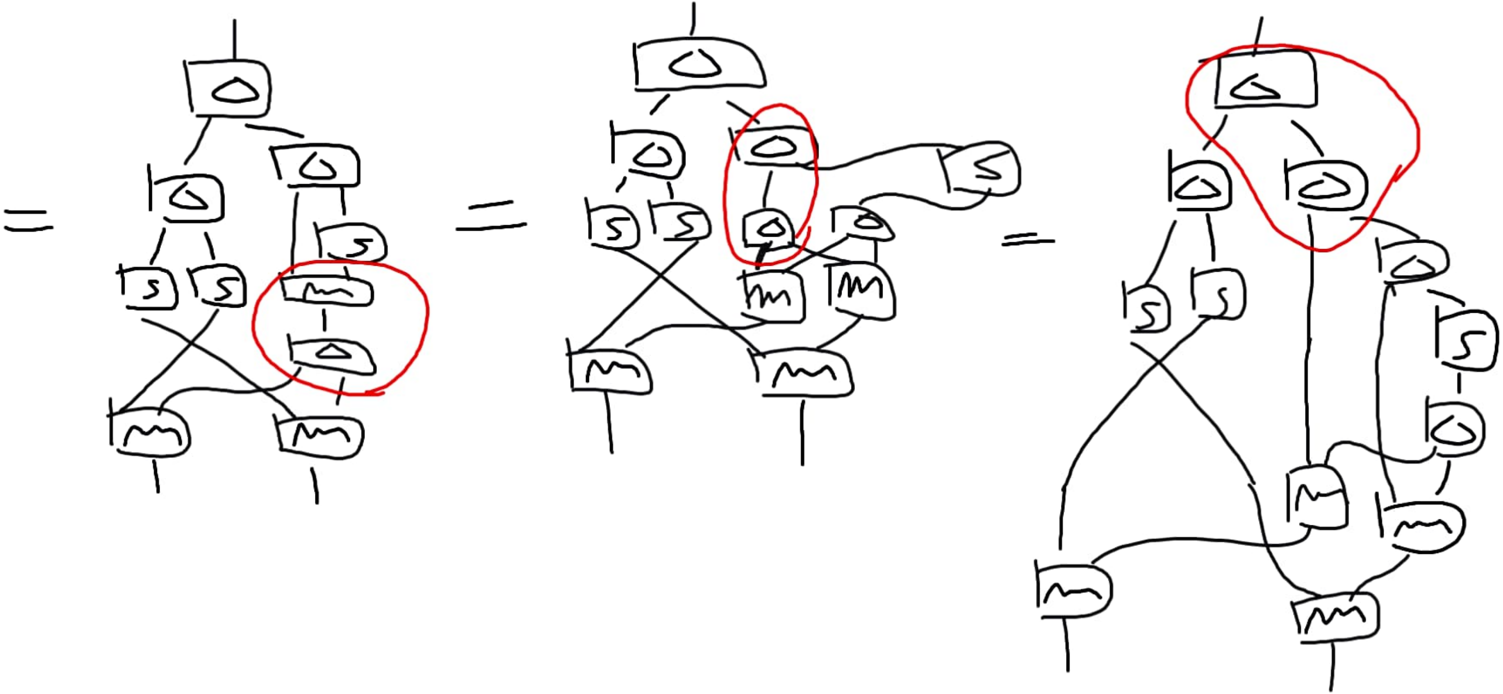


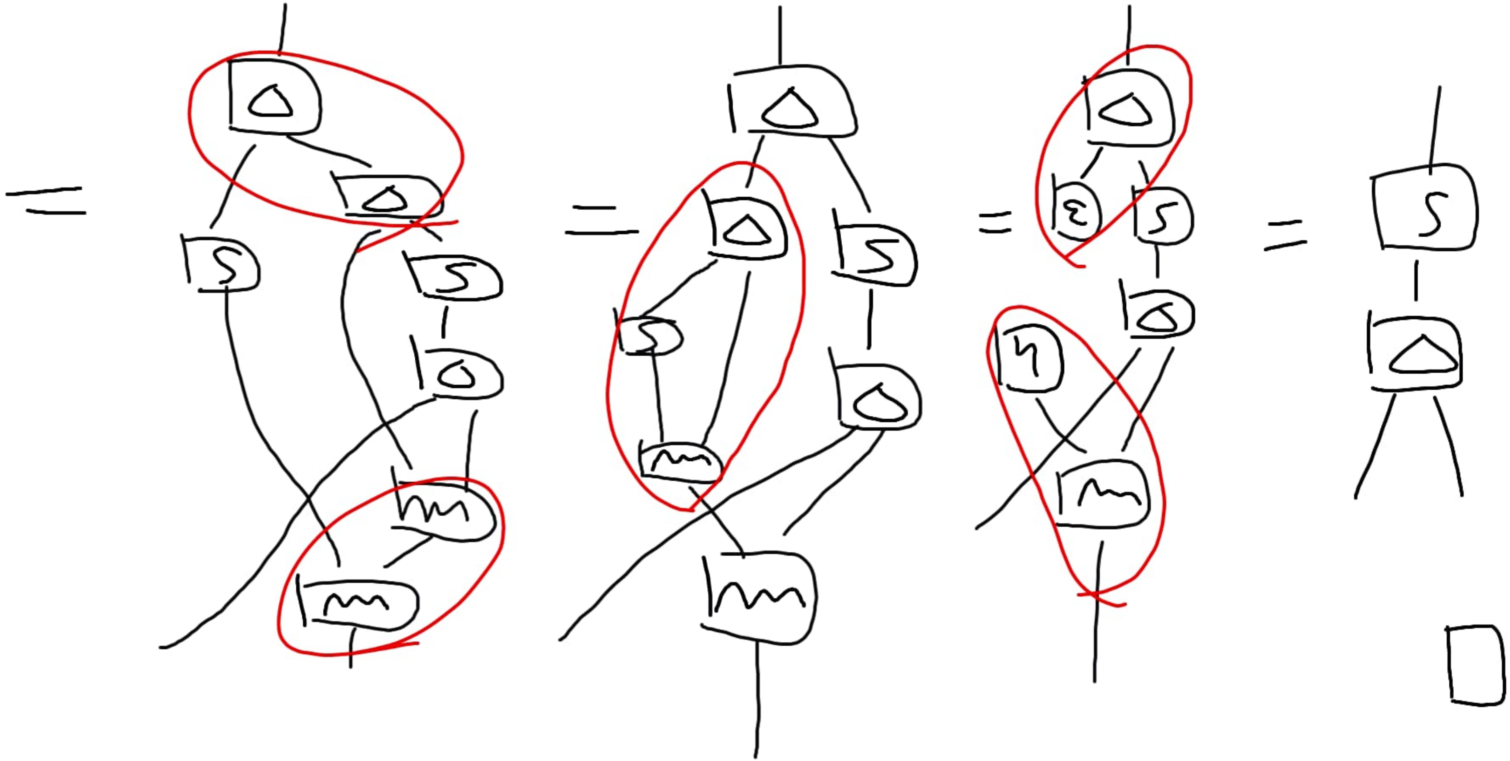
□

Pf of lem 4.

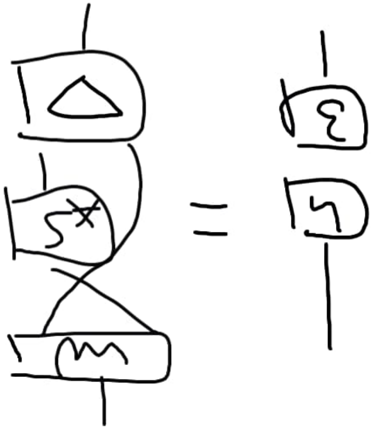




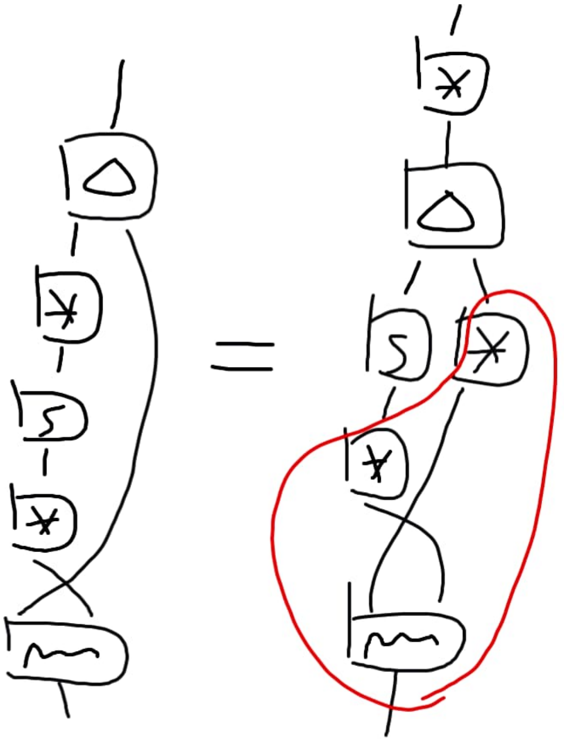


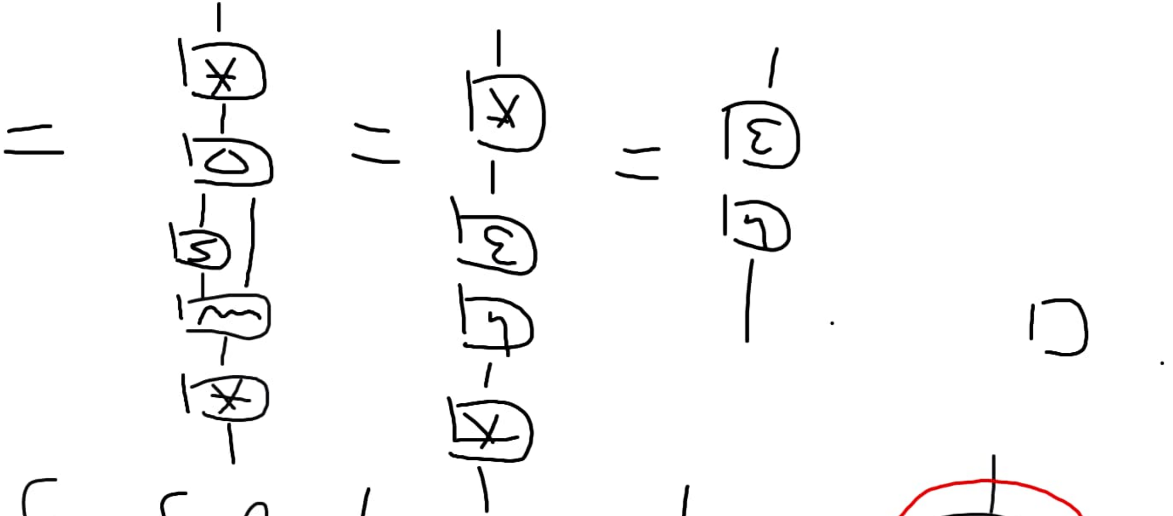


Pf of lem5



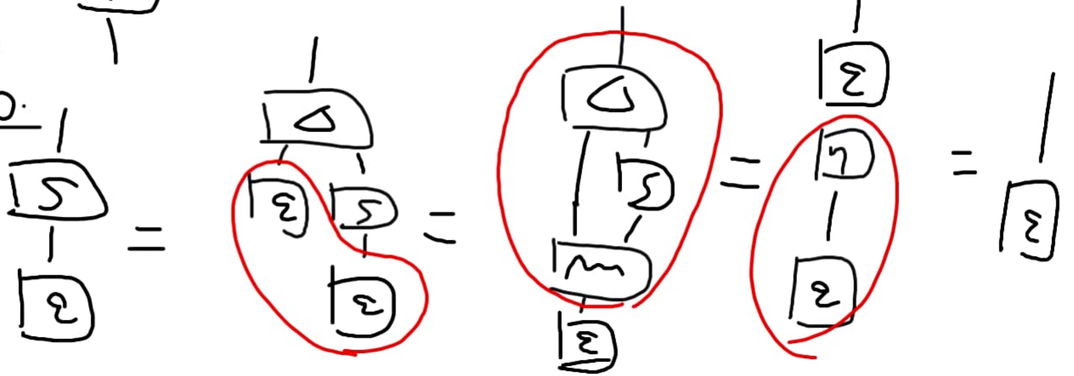
のみ示す。





Pf of lemb.

$\varepsilon \circ \delta =$

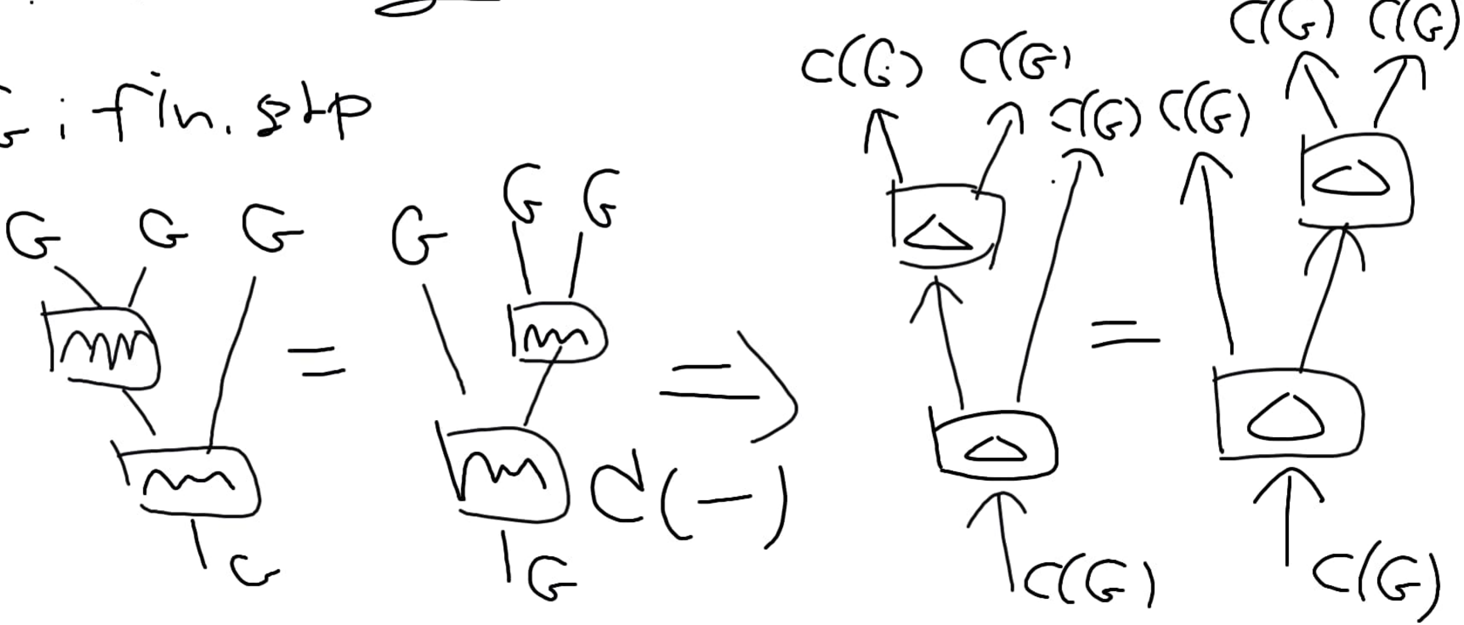


$= \varepsilon$

◎ Graphical Notationの利点 ◎

1. Hopf alg の def がよく分かる

G : fin. grp



(こっそり $C(G\times G) \underset{\text{top alg}}{\cong} C(G)\otimes C(G)$

も使った)

2. 直感的.

3. 元を見てない.

$(\longleftrightarrow$ Sweedler's notation$)$.

Sweedler's Notation

$$\Delta : A \longrightarrow A \otimes A$$

$$\Delta(a) = \sum_{i=1}^{n} x_i \otimes b_i = \underline{a_{(1)} \otimes a_{(2)}}$$

形式的な記号

$$(id \otimes \Delta)\Delta(a) = a_{(1)} \otimes \Delta(a_{(2)})$$

$$\parallel \qquad\qquad \parallel$$

$$a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$$

$$\parallel \qquad\qquad \parallel$$

$$(\Delta \otimes id)\Delta(a) = \Delta(a_{(1)}) \otimes a_{(2)}$$

$$(\varepsilon \otimes id)\Delta(a) = a \iff \varepsilon(a_{(1)})a_{(2)} = a$$

$$(id \otimes \varepsilon)\Delta(a) = a \iff a_{(1)}\varepsilon(a_{(2)}) = a$$

$$m(id \otimes S)\Delta(a) = \eta(\varepsilon(a))$$

$$\iff a_{(1)}S(a_{(2)}) = \eta(\varepsilon(a))$$

$$m(S \otimes id)\Delta(a) = \eta(\varepsilon(a))$$

$$\iff S(a_{(1)})a_{(2)} = \eta(\varepsilon(a)).$$

My research

- (L)CQG : (局所)コンパクト量子群

  ─ (局所)コンパクト(リー)群を変形したもの

具体的には LCQG $SL_q(3,\mathbb{R})$ の構成と irreps の分類

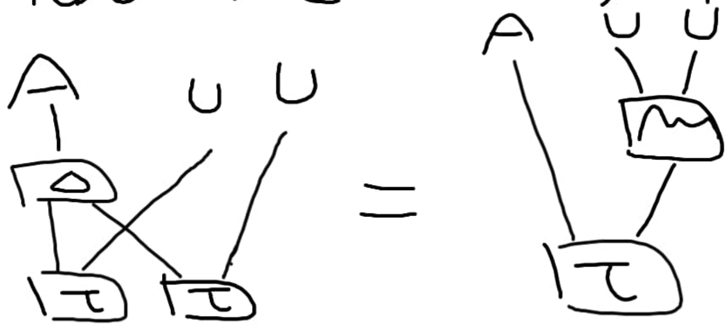
手法：Drinfeld double の活用

Drinfeld double とは？

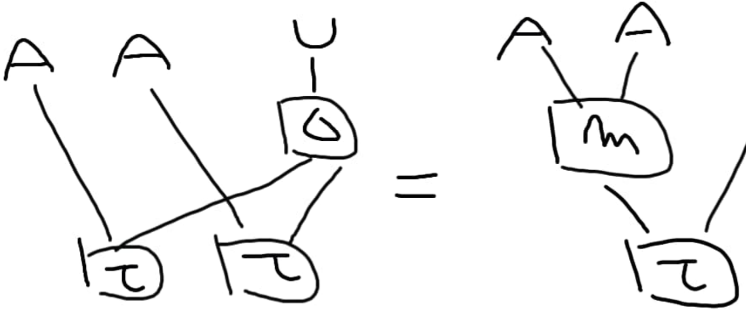
A, U : Hopf $*$-algs

$\exists \tau : A \otimes U \to \mathbb{C}$ unitary pairing

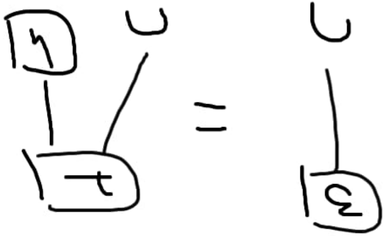
i.e.



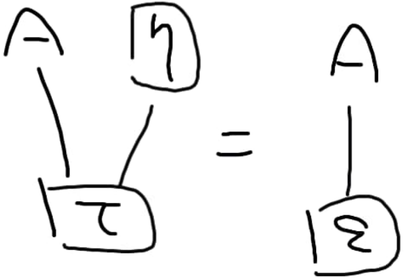
(τ1)



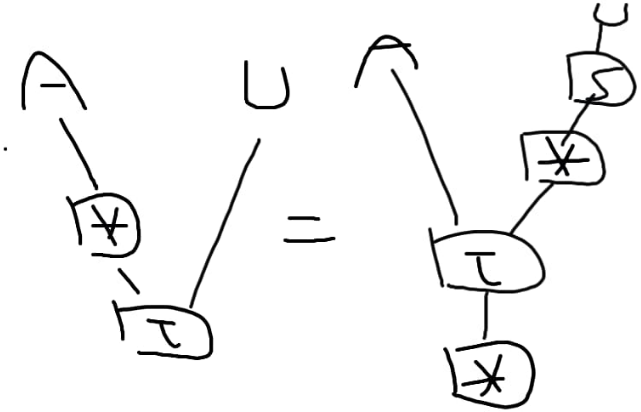
(τ2)



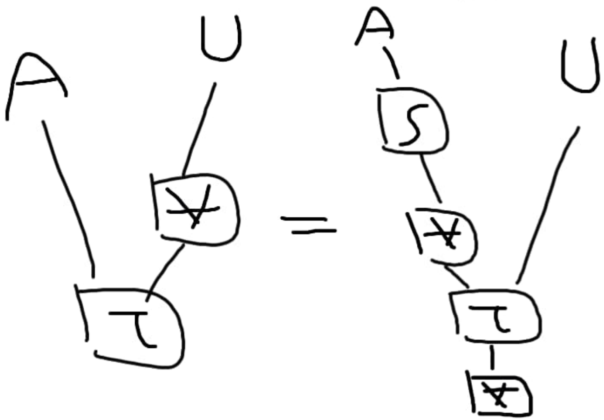
(τ3)



(τ4)



(τ5)



(τ6)

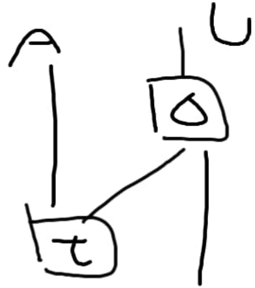
· $I \subset U$ unital left coideal $*$-subalg

$\Delta(I) \subset U \otimes I$

· $(a \otimes h \mapsto a \triangleright h) \overset{\text{def}}{=}$ 

$(a \otimes h \mapsto h \triangleright a) \overset{\text{def}}{=}$ 

$$(a \otimes h \mapsto h \triangleleft a) \overset{\text{def}}{=}$$



$$(a \otimes h \mapsto a \triangleleft h) \overset{\text{def}}{=}$$

$$I^{\perp} \underset{\text{def}}{=} \{ a \in A \mid \forall h \in I, \ h \triangleright a = \varepsilon(h)\, a \}$$

Def $I \subset U$ as above

$J \subset I^{\perp} \subset A$ unital ***-subalg** <u>right coideal</u> $\Delta(J) \subset J \otimes A$

Drinfeld double

$A \bowtie U \overset{\text{def}}{=}$ the unital $*$-alg gen by A & U ($D(A, U)$) with the rels

$$ha = (S^{-1}(h_{(3)}) \triangleright a \triangleleft h_{(1)})\, h_{(2)}$$

$$ah = (a_{(1)} \triangleright h \triangleleft S^{-1}(a_{(3)}))\, a_{(2)}$$

$J \bowtie I \overset{\text{def}}{=}$ the unital $*$-subalg of $A \bowtie U$ gen by J & I.

$(\mathcal{D}(J, I))$

◎ Drinfeld double を使って… ◎

$$U_q(\mathfrak{sl}(3, \mathbb{R})) \overset{\text{def}}{=} U_q(\mathfrak{k}_0)^{\perp} \bowtie U_q^{\mathfrak{k}_0}$$

($U_q(\mathfrak{k}_0)$: $\mathfrak{su}(3)$ のある Lie subalg の変形)

そして、$U_q(\mathfrak{sl}(3, \mathbb{R}))$-moduleの分類を。

References

- G. Kuperberg, Involutory Hopf Algebras and 3-Manifold Invariants
- M. Takeuchi, Free Hopf algebras generated by coalgebras